Some more Problems

1. Let $G = \{(a,b) : a,b \in \mathbb{Q}, a \neq 0\}$. Define the binary operation $*$ on G by $(a,b)*(c,d) = (ac, ad+b)$. Show that $(G,*)$ is a group. Is it abelian?

Solution: Let (a,b), (c,d) and $(e,f) \in G$

Now $[(a,b)*(c,d)]*(e,f) = (ac, ad+b)*(e,f) = (ace, acf+ad+b)$

Also $(a,b)*[(c,d)*(e,f)] = (a,b)*(ce, cf+d) = (ace, acf+ad+b)$

So, $[(a,b)*(c,d)]*(e,f) = (a,b)*[(c,d)*(e,f)]$. So, $*$ is associative in G.

$(1,0)$ is the identity in G as $(a,b)*(1,0) = (1,0)*(a,b) = (a,b)$, $\forall (a,b) \in G$.

and the inverse of (a,b) is $(1/a, -b/a)$ as

$$(a,b)*\left(\frac{1}{a}, -\frac{b}{a}\right) = (1,0) = \left(1/a, -b/a\right)*(a,b).$$

So, $(G,*)$ is a group.

G is not abelian as

$$(1,2)*(3,4) = (3, 4+2) = (3,6)$$

and $(3,4)*(1,2) = (3, 6+4) = (3,10)$

So, $(1,2)*(3,4) \neq (3,4)*(1,2)$

2. Choose the correct alternative. Justify your answer

(a) The centre of the group S_3 is

(i) $\{\rho_0, \rho_3\}$ (ii) $\{\rho_0, \rho_1, \rho_2\}$ (iii) $\{\rho_0, \rho_4\}$ (iv) None of the above

(b). The solution of the equation $(12)x = (23)$ in S_3 is

(i) (132) (ii) (13) (iii) (123) (iv) None of the above

(c) The centre of the group $GL(2,\mathbb{R})$ is

(i) $\left\{\begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{R}, a \neq 0\right\}$ (ii) $\left\{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{R}, a \neq 0\right\}$ (iii) $\left\{\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} : a \in \mathbb{R}, a \neq 0\right\}$ (iv) None of the above

3. Let G be a finite group whose order is not divisible by 3. Suppose $(ab)^3 = a^3b^3$, for all $a, b \in G$, then show that G is abelian.

Solution: Let $a, b \in G$ Then

$$(ab)^3 = a^3b^3$$
$$\Rightarrow ababab = a^3b^3$$
$$\Rightarrow baba = a^2b^2 \text{ (Cancellation Laws)}$$
$$\Rightarrow (ba)^2 = a^2b^2 \quad \ldots \quad (1)$$

Again as $(ba)^3 = b^3a^3$

$$\Rightarrow ba(ba)^2 = b^3a^3$$
$$\Rightarrow (ba)\cancel{b}a^2b^2 = b^3a^3 \quad [\text{using } (1)]$$
$$\Rightarrow a^3b^2 = b^2a^3 \quad \ldots \quad (2)$$

Consider now
$$(a^{-1}b^{-2}ab^2)^3 = (a^{-1})^3(b^{-2}ab^2)^3$$
$$= a^{-3}(b^{-2}ab^2)^3$$
$$= a^{-3}(b^{-2}a^3b^2) \quad [\because (x^{-1}ax)^n = x^{-1}a^nx \text{ for integer } n]$$
$$= a^{-3}(b^{-2}b^2a^3) \quad [\text{from } (2)]$$
$$= a^{-3}a^3 = e, \text{ } e \text{ is the identity element in } G.$$

$$\Rightarrow o(a^{-1}b^{-2}ab^2) \text{ divides } 3$$

So, $o(a^{-1}b^{-2}ab^2) = 1$ or 3

If $o(a^{-1}b^{-2}ab^2) = 3$, then 3 ~~divide~~ divides $o(G)$, a contradiction.

So, $o(a^{-1}b^{-2}ab^2) = 1 \Rightarrow a^{-1}b^{-2}ab^2 = e$

$$\Rightarrow ab^2 = b^2a \quad \ldots \quad (3)$$

Again, from (1) $(ba)^2 = a^2b^2 = a(ab^2) = a(b^2a)$ [from (3)]

$$\Rightarrow (ba)(ba) = ab^2a$$
$$\Rightarrow bab = ab^2$$
$$\Rightarrow ba = ab$$

So, G is abelian.

4. If N is a normal subgroup of order 2, of a group G then show that $N \subseteq Z(G)$, the centre of G.

Solution: Let $N = \{a, e\}$, e is the identity element in G. Since $e \in Z(G)$ (centre being a subgroup of G), all that we have to show is that $a \in Z(G)$

Let $x \in G$, then as $a \in N$ and N is normal, $xax^{-1} \in N$

$\Rightarrow xax^{-1} = a$ or $xax^{-1} = e$

Since $xax^{-1} = e \Rightarrow xa = x \Rightarrow a = e$, a contradiction, we have $xax^{-1} = a \Rightarrow xa = ax \Rightarrow a \in Z(G)$

So, $N \subseteq Z(G)$.

5. Show that a subgroup N of a group G is normal if and only if $xy \in N \Rightarrow yx \in N$, for all $x, y \in G$.

Solution: Let N be normal in G and let $xy \in N$

Since $yx = y(xy)y^{-1}$ and $xy \in N$, $y \in G$, N is normal in G we find $y(xy)y^{-1} \in N \Rightarrow yx \in N$

~~Conversely~~ Conversely, let $n \in N$ and $x \in G$.

Then $n \in N \Rightarrow (nx)x^{-1} \in N$

$\Rightarrow x^{-1}(nx) = x^{-1}nx \in N$ (As $xy \in N \Rightarrow yx \in N$)

$\Rightarrow N$ is normal in G.

6. Give an example of an infinite group which has a subgroup having finite index.

Solution: Consider the group $(\mathbb{Z}, +)$ and consider the subgroup $3\mathbb{Z}$. Then $0+3\mathbb{Z}$, $1+3\mathbb{Z}$ and $2+3\mathbb{Z}$ are three distinct left cosets of the normal subgroup $3\mathbb{Z}$.

So, the index is 3 which is finite.

7. Let G be a non-abelian group of order pq where p, q are primes then prove that $o(Z(G))=1$

Solution: Since G is non-abelian, $G/Z(G)$ is not cyclic (Problem 19, Page-62)

Now $o(Z(G))$ divides $o(G)$

So, $o(Z(G))$ divides pq

$\Rightarrow o(Z(G)) = \cancel{1, p, q, pq}$ 1 or p or q or pq

$o(Z(G)) = pq \Rightarrow Z(G) = G \Rightarrow G$ is abelian, a contradiction.

$o(Z(G)) = p \Rightarrow o\left(G/Z(G)\right) = \frac{pq}{p} = q$, a prime. So $G/Z(G)$ is cyclic, a contradiction. Similarly $o(Z(G)) \neq q$

So, $o(Z(G)) = 1$

8. Let G be the group of all non-zero complex numbers under multiplication and let G' be the group of all real 2×2 matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, where not both a and b are zero, under multiplication. Show that G is isomorphic to G'

Solution: Define a map $\phi: G \to G'$ by

$$\phi(a+ib) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

ϕ is clearly well defined.

Also $\phi(a+ib) = \phi(c+id) \Rightarrow \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$

$$\Rightarrow a = c, \quad b = d$$

$$\Rightarrow a+ib = c+id$$

So ϕ is injective. By definition, ϕ is surjective.

Now $\phi(a+ib)\phi(c+id) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} ac-bd & ad+bc \\ -ad-bc & ac-bd \end{bmatrix}$

and $\phi((a+ib)(c+id)) = \phi((ac-bd)+i(ad+bc)) = \begin{bmatrix} ac-bd & ad+bc \\ -ad-bc & ac-bd \end{bmatrix}$

Hence ϕ is a homomorphism.

So, ϕ is an isomorphism. So, G and G' are isomorphic.

7. Let G be the group $(\mathbb{R}, +)$ and N is the subgroup $(\mathbb{Z}, +)$ of G. Prove that G/N is isomorphic to the group H of all complex numbers of absolute value 1 under multiplication.

Solution: Define a map $f: G \to H$ by

$$f(x) = e^{2\pi x i}, \quad x \in G. \quad \left(\text{here } \left|e^{2\pi x i}\right| = 1\right)$$

Let $h \in H$ and let $h = a+ib$ such that $|a+ib| = 1$

So, we can write $h = \cos\theta + i\sin\theta = e^{i\theta}$ for some $\theta \in G$.

$$\text{So, } f\left(\frac{\theta}{2\pi}\right) = e^{2\pi\frac{\theta}{2\pi}i} = e^{i\theta} = h$$

So, f is onto or surjective.

Again $\cancel{f(x_1+x_2) = e}$ $f(x_1+x_2) = e^{2\pi(x_1+x_2)i}$ $(x_1, x_2 \in \mathbb{R})$

$$= e^{2\pi x_1 i} \cdot e^{2\pi x_2 i}$$

$$= f(x_1) f(x_2)$$

Hence f is a homomorphism

So, f is an onto homomorphism

Here

$$\begin{aligned}
\ker f &= \{x \in \mathbb{R} : e^{2\pi x i} = 1\} \\
&= \{x \in \mathbb{R} : \cos 2\pi x + i\sin 2\pi x = 1\} \\
&= \{x \in \mathbb{R} : \cos 2\pi x = 1, \sin 2\pi x = 0\} \\
&= \{x \in \mathbb{R} : 2\pi x = 2\pi n, n \in \mathbb{Z}\} \\
&= \mathbb{Z} = N \text{ (here)}
\end{aligned}$$

So, by first Isomorphism theorem, we get G/N is isomorphic to H.