Some Problems

1. Show that, if in a finite semi-group G, $xy = zx \Rightarrow y = z$, for any $x,y,z \in G$, then G is an abelian group.

Proof: Let $x, y \in G$ Then $x(yx) = (xy)x \Rightarrow yx = xy$ by the property $xy = zx \Rightarrow y = z$, for any $x, y, z \in G$.

So, G is abelian. Now as G is abelian

$xy = zx$ can be written as $xy = xz$ or $yx = zx$

So, the property $xy = zx \Rightarrow y = z$ gives both the cancellation laws.

So, G is a finite semi-group where both the cancellation laws hold. So, G is a Group.

Hence G is an abelian group.

2. Choose the correct alternative. Justify your answer

(a) Let $a, b, c \in G$, G is a group. A solution of the equation of the equation $axb = c$ is

(i) $b^{-1}ca^{-1}$ (ii) $a^{-1}c^{-1}b$ (iii) $a^{-1}cb^{-1}$ (iv) $a^{-1}c^{-1}b^{-1}$

(b) Let $a, b \in G$, G be a group. A solution of the equation

$xabx^{-1} = ba$ is

(i) ab^{-1} (ii) ba (iii) b (iv) a

Ans: Exercise

3. If in a group G, $axb = cxd \Rightarrow ab = cd$ for any $a, b, c, d, x \in G$. Prove that G is abelian.

Ans: ~~Observe that~~ Let $a, b \in G$. Observe that

$aa^{-1}b = ba^{-1}a$. This implies $ab = ba$ as $axb = cxd \Rightarrow ab = cd$. So, G is abelian.

4. Give an example of a non-cyclic group whose proper subgroups are all cyclic.

Ans. Exercise

5. Let G be an abelian group and n be a fixed positive integer. Show that $H = \{a \in G : o(a) \text{ divides } n\}$ is a subgroup of G.

Proof: $o(e) = 1$ and 1 divides n. So $e \in H$ and $H \neq \phi$.

Let $a, b \in H \Rightarrow o(a)$ divides n, $o(b)$ divides n

$\Rightarrow a^n = e, \; b^n = e$

$\Rightarrow (ab^{-1})^n = a^n (b^{-1})^n$ (As G is abelian)

or, $(ab^{-1})^n = a^n (b^n)^{-1}$

$= e \cdot e^{-1} = e$

So $o(ab^{-1})$ divides n

So, $ab^{-1} \in H$

Hence H is a subgroup of G.

6. Show that a group can not contain exactly two elements of order 2.

Ans: ~~If possible~~, let a and b are two distinct elements of order 2 in G. If a and b commute then $ab \neq a$, $ab \neq b$ and $ab \neq e$. Also $(ab)^2 = a^2 b^2 = e$

So, ab is another element of order 2. If a and b do not commute, then $ab \neq ba$.

So, $aba \neq a$, $aba \neq b$, $aba \neq e$ and

$(aba)^2 = (aba)(aba) = aba^2ba = ab^2a = a^2 = e$

So, aba is another element of order 2. So, a group can not have exactly two elements of order 2

7. If x is an element of a ~~cyclic~~ group of order 15 and exactly two of x^3, x^5 and x^9 are equal, determine $o(x^{13})$

Ans: Let $x \in G$, where G is a group of order 15

Case 1 Let $x^3 = x^5$. So, $x^2 = e$. So, $o(x)$ divides 2 and and 15. So, $o(x) = 1$. So, $x = e$ So, $x^{13} = e$

So, $o(x^{13}) = 1$

Case 2 Let $x^3 = x^9$. So, $x^6 = e$. So, $o(x)$ divides 6 and 15

So, $o(x) = 3$ So, $o(x^{13}) = o(x \cdot (x^3)^4) = o(x) = 3$

Case 3 Let $x^5 = x^9$. So, $x^4 = e$. So, $o(x)$ divides 4 and 15

So, $o(x) = 1 \Rightarrow x = e$ So, $o(x^{13}) = o(e) = 1$

8. Let G be a group of order p^n, p a prime. Show that G contains an element of order p.

Solution: Let $a \in G$, $a \neq e$. Then $H = \langle a \rangle$ is a cyclic subgroup of G. Now $o(H)$ divides $o(G) = p^n$. So, $o(H) = p^m$, $0 < m \leq n$ and $m \in \mathbb{Z}$. Hence for every divisor d of p^m, $\exists$ a subgroup of H of order p. So for p, $\exists$ a subgroup T of H such that $o(T) = p$. So, $\exists$ $b \in T$ such that $T = \langle b \rangle$ and b is of order p. Hence G contains an element of order p.

9. Express the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 8 & 5 & 6 & 4 & 7 & 1 \end{pmatrix}$ on 8 symbols 1, 2, 3, 4, 5, 6, 7, 8 as a product of disjoint cycles and then as a product of transposition. Is σ an even permutation?

Solution: We have $\sigma(1) = 2$, $\sigma^2(1) = \sigma(2) = 3$, $\sigma^3(1) = \sigma(3) = 8$, and $\sigma^4(1) = \sigma(8) = 1$. Thus $(1\ 2\ 3\ 8)$ is a cycle. Now 4 is the first element not appearing in $(1\,2\,3\,8)$. We have $\sigma(4)=5$, $\sigma^2(4)=\sigma(5)=6$ and $\sigma^3(4)=\sigma(6)=4$. Hence, $(4\,5\,6)$ is also a cycle in σ. Next, 7 is the first element not appearing $(1\,2\,3\,8)$ and $(4\,5\,6)$. Now $\sigma(7)=7$. Since all the symbols $1,2,3,4,5,6,7,8$ appear in one of the cycles $(1\,2\,3\,8)$, $(4,5\,6)$ and (7), we have $\sigma=(1\,2\,3\,8)\circ(4,5,6)$. Now $(1\,2\,3\,8)=(18)\circ(13)\circ(12)$ and $(4,5\,6)=(46)\circ(45)$. Thus $\sigma=(18)\circ(13)\circ(12)\circ(46)\circ(45)$. Since σ is a product of 5 transposition, it is not an even permutation.

10. Write all the elements of S_4. Show that S_4 has no elements of order ≥ 5.

Solution: Let $\sigma\in S_4$ and $\sigma=\sigma_1\circ\sigma_2\circ\cdots\circ\sigma_k$, a product of disjoint cycles. Since S_4 is a permutation group on the symbols $1,2,3,4$, $k\leq 2$. If $k=2$, then σ is a product of two disjoint transpositions. The number of distinct cycles of length 2 is 6, the number of distinct cycles of length 3 is 8 and number of distinct cycles of length 4 is 6. If e be the identity permutation, then we have

$$S_3=\{e,(12),(13),(14),(2,3),(24),(34),(1\,2\,3),(1\,3\,2),(2\,3\,4),(2\,4\,3),(1\,3\,4),(1\,4\,3),(1\,2\,4),(1\,4\,2),(1\,2\,3\,4),(1\,3\,2\,4),(1\,4\,2\,3),(1\,2\,4\,3),(1\,3\,4\,2),(1\,4\,3\,2),(12)\circ(34),(14)\circ(32),(13)\circ(24)\}$$

Since each 2-cycle is of order 2, each 3-cycle is of order 3, each 4-cycle is of order and the order of the product of two disjoint 2-cycles is 2, S_4 has no elements of order ≥ 5.

11. Find the order of $(1\,2\,3\,4)\circ(5\,6\,7)$ in S_7

Solution: $o((1\,2\,3\,4))=4$, $o((5\,6\,7))=3$. Now $(1\,2\,3\,4)$ and $(5\,6\,7)$ are disjoint. Hence $(1\,2\,3\,4)\circ(5\,6\,7)=(5\,6\,7)\circ(1\,2\,3\,4)$. If a and b are two elements of a group G such that $ab=ba$ such that $o(a)=m$ and $o(b)=n$ and $\gcd(m,n)=1$, Then $o(ab) = mn$. Using this result, we find that the order of $(1234)\circ(567)$ is 12.

12. Find the order of $(1234)\circ(56)$ in S_6

Solution: $o(1234) = 4$, $o(56) = 2$ Now (1234) and (56) are disjoint and so they commute. Thus, $((1234)\circ(56))^4 = e$, e is the identity permutation in S_6. Now $(1234)\circ(56) \neq e$, $((1234)\circ(56))^2 = ((1234))^2\circ((56))^2 = (1234)^2 \neq e$. If $((1234)\circ(56))^3 = e$, then the order of $(1234)\circ(5,6)$ will be 3 and 3 divides 4, a contradiction. Hence order of $(1234)\circ(56)$ is 4.

13. Let G be a group and $a, b \in G$. Suppose that $a^2 = e$ and $ab^4a = b^7$. Show that $b^{33} = e$, e is the identity element in G.

Solution: As $a^2 = e \Rightarrow a = a^{-1}$, so $ab^4a = a^{-1}b^4a = (a^{-1}ba)^4$

Now $(ab^4a)^4 = (a^{-1}ba)^{16} = a^{-1}b^{16}a = (b^7)^4 = b^{28}$

So, $a^{-1}b^{16}a = b^{28}$

or, $b^{16} = ab^{28}a^{-1} = a^{-1}b^{28}a = (a^{-1}b^4a)^7 = (b^7)^7 = b^{49}$

So, $b^{33} = e$

14. Let G be a group. If $a, b \in G$ are such that $a^4 = e$ and $a^2b = ba$, show that $a = e$, e is the identity element in G.

Solution: $a^2b = ba \Rightarrow a^4b = a^2ba$

$\Rightarrow b = a^2ba \quad [\because a^4 = e]$

$\Rightarrow b = ba^2 \quad [\because a^2b = ba]$

$\Rightarrow a^2 = e$. So $a^2b = ba \Rightarrow b = ba \Rightarrow a = e$.