Similarly, $\frac{b-a}{2} \in Q$ is ~~a solution~~ the unique solution of the equation $y \circ a = b$, $a, b \in Q$. So $(Q, \circ)$ is a quasigroup.

Now $(1 \circ 3) \circ 2 = (2+3) \circ 2 = 5 \circ 2 = 10+2 = 12$

and $1 \circ (3 \circ 2) = 1 \circ (6+2) = 1 \circ 8 = 2+8 = 10$

So $(1 \circ 3) \circ 2 \neq 1 \circ (3 \circ 2)$, So, $(Q, \circ)$ is not a semigroup.

1(b) Does there exist (i) a left identity in $(Q, \circ)$ (ii) a right identity in $(Q, \circ)$, where $(Q, \circ)$ is the groupoid as in 1(a).

Solution: (i) If $\exists$ a left identity e_1 in $(Q, \circ)$ then

$$e_1 \circ a = a, \ \forall a \in Q$$

$$\text{or, } \ 2e_1 + a = a \quad \text{or, } \ e_1 = 0$$

So, $\exists$ a left identity $0 \in Q$

(ii) If $\exists$ a right identity $e_2 \in Q$ then

$$a \circ e_2 = a, \ \forall a \in Q$$

$$\text{So, } \ 2a + e_2 = a \quad \text{or, } \ e_2 = a - 2a = -a$$

As e_2 depends on a, so, ~~∄~~ there does not exist any right identity in $(Q, \circ)$.

2. Let $\mathbb{R}^* = \mathbb{R} - \{0\}$. Define a binary operation $\circ$ on $\mathbb{R}^*$ by $a \circ b = |ab|$, $a, b \in \mathbb{R}^*$. Show that $(\mathbb{R}^*, \circ)$ is a semigroup but not a quasigroup.

Solution: Here $(\mathbb{R}^*, \circ)$ is a groupoid

Now $a \circ (b \circ c) = a \circ (|bc|) = |a|bc|| = |abc|$

and $(a \circ b) \circ c = |ab| \circ c = ||ab|c| = |abc|$

So, $(a \circ b) \circ c = a \circ (b \circ c)$, $\forall a, b, c \in \mathbb{R}^*$

Consider the equation $2 \circ x = 3$

Take $x = -\frac{3}{2}$. Then $2 \circ (-\frac{3}{2}) = |2(\frac{-3}{2})| = 3$

Take $x = \frac{3}{2}$ Then $2 \circ \frac{3}{2} = |2 \cdot \frac{3}{2}| = 3$

$2 \circ x = 3$ ~~has no unique~~ does not have unique solution in $\mathbb{R}^*$

So, $(\mathbb{R}^*, \circ)$ is not a quasigroup.

3. Let M ~~be the set of all~~ be the set of all real matrices $\left\{ \begin{pmatrix} a & a \\ b & b \end{pmatrix} : a+b \neq 0 \right\}$. Prove that

(i) $(M, \cdot)$ is a semi-group where '$\cdot$' is matrix multiplication

(ii) there is no left identity in the semigroup

(iii) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is a right identity

Solution: Let $A = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$ $B = \begin{pmatrix} c & c \\ d & d \end{pmatrix}$ where a, b, c, d are real nos.

and $a+b \neq 0$ and $(c+d) \neq 0$

Then $AB = \begin{pmatrix} ac+ad & ac+ad \\ bc+bd & bc+bd \end{pmatrix} = \begin{pmatrix} a(c+d) & a(c+d) \\ b(c+d) & b(c+d) \end{pmatrix}$

$= \begin{pmatrix} e & e \\ f & f \end{pmatrix}$ where $e = a(c+d)$ and $f = b(c+d)$

and $e+f = a(c+d) + b(c+d) = (a+b)(c+d)$

$\neq 0$ as $a+b \neq 0$ and $c+d \neq 0$

So $(M, \cdot)$ is a groupoid

As Matrix multiplication is associative, $(M, \cdot)$ is a semigroup.

(ii) If $\exists$ a left identity $E = \begin{pmatrix} c & c \\ d & d \end{pmatrix}$ in $(M, \cdot)$ then for any $A = \begin{pmatrix} a & a \\ b & b \end{pmatrix} \in M$

$EA = A$ So $\begin{pmatrix} c(a+b) & c(a+b) \\ d(a+b) & d(a+b) \end{pmatrix} = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$

$\Rightarrow$ So, $c(a+b) = a$ and $d(a+b) = b$

So, $c = \frac{a}{a+b}$ and $d = \frac{b}{a+b}$ as $a+b \neq 0$

Also $c+d = 1$ but c, d both depend on a, b

So, there exists no left identity in the semigroup $(M, \cdot)$.

Now let $A = \begin{bmatrix} a & a \\ b & b \end{bmatrix} \in M$ then $a+b \neq 0$

$$\cancel{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & a \\ b & b \end{bmatrix} = \begin{bmatrix} a+b & a+b \\ 0 & 0 \end{bmatrix}}$$

Now $A \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & a \\ b & b \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & a \\ b & b \end{bmatrix}$

So, $A \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = A$, for all $A \in M$ and $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in M$. So, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is a right identity in M

4. Define a binary operation $*$ on the set $\mathbb{Z} \times \mathbb{Z}$ by $(a,b) * (c,d) = (a+c, a+d)$, $(a,b), (c,d) \in \mathbb{Z} \times \mathbb{Z}$. Prove that $(\mathbb{Z} \times \mathbb{Z}, *)$ is a semigroup but not a monoid.

Solution: $(\mathbb{Z}, *)$ is a groupoid

Now let $(a,b), (c,d), (e,f) \in \mathbb{Z} \times \mathbb{Z}$

now $(a,b) * \big((c,d) * (e,f)\big) = (a,b) * (c+e, c+f) = (a+c+e, a+c+f)$

$\big((a,b) * (c,d)\big) * (e,f) = (a+c, a+d) * (e,f) = (a+c+e, a+\cancel{c}\ a+c+f)$

So, $(\mathbb{Z}, *)$ is a semigroup.

If $\exists$ an identity $e = (e_1, e_2) \in \mathbb{Z} \times \mathbb{Z}$

then $e * (a,b) = (a,b) * e = (a,b)$, $\forall (a,b) \in \mathbb{Z} \times \mathbb{Z}$

So, Now $e * (a,b) = (e_1, e_2) * (a,b) = (e_1 + a, e_1 + b)$

So, $e * (a,b) = (a,b) \Rightarrow e_1 + a = a$, $e_1 + b = b$

$\Rightarrow e_1 = 0$

Also $(a,b) * e = (a,b) * (e_1, e_2) = (a+e_1, a+e_2)$

So, $(a+e_1, a+e_2) = (a,b) \Rightarrow a + e_1 = a$, $a + e_2 = b$

$\Rightarrow e_1 = 0$ and $e_2 = b - a$

So, e depends on (a,b)

So, there exists no identity in $(\mathbb{Z}\times\mathbb{Z}, *)$

So, $(\mathbb{Z}\times\mathbb{Z}, *)$ is a semigroup but not a monoid.

5. Prove that $(\mathbb{Z}\times\mathbb{Z}, *)$ is commutative monoid where $*$ is defined by $(a,b)*(c,d) = (ac, bd)$, $(a,b),(c,d) \in \mathbb{Z}\times\mathbb{Z}$. Find the units in the monoid. Find the idempotent elements in the monoid.

Solution: $(\mathbb{Z}\times\mathbb{Z}, *)$ is a groupoid.

Now $(a,b)*\left((c,d)*(e,f)\right) = (a,b)*(ce, df) = (ace, bdf)$

and $\left((a,b)*(c,d)\right)*(e,f) = (ac,bd)*(e,f) = (ace, bdf)$

for $(a,b),(c,d),(e,f) \in \mathbb{Z}\times\mathbb{Z}$

So, $(\mathbb{Z}\times\mathbb{Z}, *)$ is a semigroup

Now $(1,1) \in \mathbb{Z}\times\mathbb{Z}$ and $(a,b)*(1,1) = (1,1)*(a,b) = (a,b)$, $\forall (a,b) \in \mathbb{Z}\times\mathbb{Z}$

So, $(1,1)$ is the identity element in $(\mathbb{Z}\times\mathbb{Z}, *)$

So, $(\mathbb{Z}\times\mathbb{Z}, *)$ is a monoid

Let (a,b) be a unit in $(\mathbb{Z}\times\mathbb{Z}, *)$. Then $\exists$ $(c,d) \in \mathbb{Z}\times\mathbb{Z}$ such that

$$(a,b)*(c,d) = (c,d)*(a,b) = (1,1)$$

or, $ac = 1$, $bd = 1$ $\Rightarrow c = \frac{1}{a}$ if $a \neq 0$

and $d = \frac{1}{b}$ if $b \neq 0$

So, if $(a,b) \in \mathbb{Z}\times\mathbb{Z}$ such that $a \neq 0$, $b \neq 0$

then (a,b) is a unit in $(\mathbb{Z}\times\mathbb{Z}, *)$

Let (a,b) be an ~~idem~~ idempotent element in $(\mathbb{Z}\times\mathbb{Z}, *)$.

$(a,b)*(a,b) = (a,b) \Rightarrow a^2 = a,\ b^2 = b \Rightarrow a = 0,1,\ b = 0,1$. So, $(0,0),(0,1),(1,0)$ and $(1,1)$ are the idempotent elements in $(\mathbb{Z}\times\mathbb{Z}, *)$