Notes on Mathematical Logic, (SEC 2) by SB (Subhasundar Bandyopadhyay)

Book followed : 1. Introduction to Mathematical Logic - Elliot Mendelson

2. Sets and Mathematical Logic - Biswadip Pal (Recently published by Techno world)

Introduction : The logic is the subject of studying reasonings and its general structures. The logic that Mathematicians use to prove their theorems is itself ~~as~~ a part of mathematics. It is a part in the same way that analysis, algebra and geometry are parts of Mathematics.

Propositions : A proposition (or, Statement) is a declarative sentence that is either true or false (but not both). Some examples are :

1. APJ Abdul Kalam was the president of India
2. $3+4=7$

Here (1) is true and (2) is false.

Propositions are denoted by capital letters A, B, ... etc. (some author also prefers small letters $p, q, r, \ldots$ etc)

Propositions may be combined in various ways to form more complicated propositions. We shall consider only truth-functional combinations of propositions, that is, ~~a~~ combinations in which the truth or falsity of the new propositions is determined by the truth or falsity of the component propositions.

The truth value of a proposition is denoted by T when it is true and denote it by F when it is false.

Truth Table: When combining propositions to form a new proposition, we find ~~the~~ the truth functional character of the new proposition for each truth value of the component proposition and write it in a table, which is called a truth table.

Negation: Negation is a ~~unitary~~ unary operation on propositions, i.e., it is a operation on a single proposition. The symbol for negation is $\neg$ (or $\sim$). Thus if A is a proposition then $\neg A$ denotes the negation of A. The truth table for $\neg A$ is given in the following table

A	$\neg A$
T	F
F	T

When A is true, $\neg A$ is False; when A is false, $\neg A$ is true.

conjunction: Another common truth-functional operation is the conjunction: "and". It is a binary operation. Conjunction of two propositions A and B, denoted by $A \wedge B$, has the following truth table:

A	B	$A \wedge B$
T	T	T
F	T	F
T	F	F
F	F	F

$A \wedge B$ is true when and only when both A and B are true. A and B are called conjuncts of $A \wedge B$. Note that there are four rows in the table, corresponding to the number of possible assignments of truth values ~~of~~ to A and B.

disjunction: Another binary operation is the disjunction: "or". It is the inclusive 'or' which is denoted by $A \vee B$, for two propositions A and B. It has the following truth table

A	B	$A \vee B$
T	T	T
F	T	T
T	F	T
F	F	F

$A \vee B$ is false when and only when both A and B are false. A and B are called disjuncts.

Implication (or conditional): Another important truth-functional operation is the implication: "if A, then B" denoted by $A \Rightarrow B$ (or $A \to B$). A is called antecedent and B is called consequent. Surely "if A then B" is false when A is true and B is false but for other cases there is no well-defined truth values. The truth table for $A \Rightarrow B$ is taken as follows:

A	B	$A \Rightarrow B$
T	T	T
F	T	T
T	F	F
F	F	T

Biconditional: Let us denote "A if and only if B" by $A \Leftrightarrow B$ (or $A \leftrightarrow B$). Such an expression is called biconditional. Clearly $A \Leftrightarrow B$ is true when and only when A and B have the same truth value.

Its truth table is given as follows:

A	B	$A \Leftrightarrow B$
T	T	T
F	T	F
T	F	F
F	F	T

Converse, contrapositive and inverse propositions:

Let A, B be two propositions and $A \Rightarrow B$ be the implication.

The converse of the conditional proposition $A \Rightarrow B$ is defined as $B \Rightarrow A$

The contrapositive of $A \Rightarrow B$ is defined as $\neg B \Rightarrow \neg A$

The inverse of $A \Rightarrow B$ is defined as $\neg A \Rightarrow \neg B$

Examples State the converse, contrapositive and inverse of these conditional statements.

(i) If it rains tonight then I will stay at home

(ii) I reached the station and the train left.

(iii) A necessary condition for a positive integer p to be prime is that either p is odd or p = 2

Ans:

(i) The proposition can be written as $A \Rightarrow B$ where

A: It rains tonight

B: I will stay at home

So, the converse is $B \Rightarrow A$: If I will stay at home then it rains tonight.

The contrapositive is $\neg B \Rightarrow \neg A$: If I will not stay at home then it does not rain tonight.

The inverse is $\neg A \Rightarrow \neg B$: If it does not rain tonight, I will not stay at home.

(ii) Here the proposition is $A \Rightarrow B$ where

A: I reached the station

B: The train left

The converse is $B \Rightarrow A$: The train left and I reached the station.

The contrapositive is $\neg B \Rightarrow \neg A$: The train did not leave and I did not reach the station

The inverse is $\neg A \Rightarrow \neg B$: I did not reach the station and the train did not leave.

Treat (iii) as Exercise

Some more examples : 1. Find the truth table for $(((\neg A) \vee B) \Rightarrow C)$

The required truth table is

A	B	C	$(\neg A)$	$((\neg A) \vee B)$	$(((\neg A) \vee B) \Rightarrow C)$
T	T	T	F	T	T
F	T	T	T	T	T
T	F	T	F	F	T
F	F	T	T	T	T
T	T	F	F	T	F
F	T	F	T	T	F
T	F	F	F	F	T
F	F	F	T	T	F

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2. Find the truth table of

(i) $((A \Rightarrow B) \vee (\neg A))$ (ii) $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$

Ans: Exercise

NOTE: $\neg, \wedge, \vee, \Rightarrow$ and $\Leftrightarrow$ are called propositional connectives or logical operators

NOTE: Let $\oplus$ denotes the exclusive use of 'or'. Then $A \oplus B$ stands for A or B ~~both~~ but not both (when A and B are two propositions). The truth table for $A \oplus B$ is

A	B	$A \oplus B$
T	T	F
F	T	T
T	F	T
F	F	F

Precedence of logical operators: The following order is the decreasing order of the strength of the connectives: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$. That is, first apply $\neg$, then apply $\wedge$, then apply $\vee$, then apply $\Rightarrow$ and lastly apply $\Leftrightarrow$. This order is used to decrease or restore the brackets in a combined propositions. For example,

$(A \Leftrightarrow (((\neg B) \vee C) \Rightarrow A))$ can be written as $A \Leftrightarrow (\neg B) \vee C \Rightarrow A$ (decreasing brackets)

Similarly $A \Leftrightarrow (\neg B) \vee C \Rightarrow A$ can be written as $(A \Leftrightarrow (((\neg B) \vee C) \Rightarrow A))$ (by restoring the brackets using the precedence of the operators)

~~Some exercises are given to ...~~

Some more exercise: 1. construct a truth table for $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$

2. construct the truth table for $((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$

3. Using the statements A: Ram is tall and B: Ram is happy, write the following statements in symbolic form:

(i) Ram is short but happy (ii) Ram is neither tall nor happy (iii) Ram is tall or unhappy (iv) Ram is short or he is both tall and unhappy

4. Let the two propositions be A: It is cold and B: It is raining. Write the statement against the following symbol:

(i) $B \wedge \neg A$ (ii) $A \vee B$ (iii) $\neg (A \vee B)$