Formal Language: A formal language is defined by an alphabet and formation rules. Alphabet of a formal language is a set of symbols on which this language is built. Some of the symbols in an alphabet may have a special meaning. The formation rules specify which strings of symbols count as well-formed. The well-formed strings of symbols are also called words, expressions, formulas or terms. Propositional Logic is a formal language

The formal languages Propositional Logic and Predicate Logic are the objects of our study and so, they are called object languages. When we study object language, we need the help of a known language to study the object language. This language is called Metalanguage.

Formal theory: The construction of a formal theory begins with the specification of an alphabet, i.e., a finite set of formal symbols and the set of all strings formed from the alphabet (string is defined to be a finite sequence of elements of the alphabet).

A formal theory has the following properties:

1. The notion of statement is effective, i.e., there is an effective procedure for deciding whether or not a string is a statement.
2. The notion of axioms is effective, i.e., there is an effective procedure for deciding whether or not a statement is an axiom.
3. The notion of logical inference is effective, i.e., given a finite sequence $S_1, S_2, \ldots, S_K$ of statements, there is an effective procedure for deciding whether or not S_K is inferred from one or more of $S_1, \ldots, S_{K-1}$, by a rule of inference

Because the notions of axiom and logical inference, the notion of proof is effective. That is, There is an effective procedure for deciding whether or not a sequence of statements is a proof. The prime purpose of a formal theory is to make the notion of proof effective.

A statment form is an expression built up from the statement letters A, B, C and so on by appropriate application of the propositional connectives.

1. All statement letters and such letters with numerical subscripts are statement forms.
2. If $\mathcal{B}$ and $\mathcal{C}$ are statement forms, then so are $(\neg\mathcal{B})$, $(\mathcal{B}\wedge\mathcal{C})$, $(\mathcal{B}\vee\mathcal{C})$, $(\mathcal{B}\Rightarrow\mathcal{C})$ and $(\mathcal{B}\Leftrightarrow\mathcal{C})$
3. Only those expressions are statement forms that are determined to be so by means of conditions 1 and 2.

Some examples of statement forms are E, $(\neg C_2)$, $(D_3\wedge(\neg B))$, $(((\neg B_1)\vee B_2)\Rightarrow(A_1\wedge C_2))$ and $(((\neg A)\Leftrightarrow A)\Leftrightarrow(C\Rightarrow(B\vee C)))$

The truth table for the statement form $((A\Leftrightarrow B)\Rightarrow((\neg A)\wedge B)$ is as follows:

A	B	$(A\Leftrightarrow B)$	$(\neg A)$	$((\neg A)\wedge B)$	$((A\Leftrightarrow B)\Rightarrow((\neg A)\wedge B))$
T	T	T	F	F	F
F	T	F	T	T	T
T	F	F	F	F	T
F	F	T	T	F	F

<u>Tautology</u>: A statement form that is always true no matter what the truth values of its statement letters may be, is called a tautology. A statement form is a tautology if and only if its corresponding truth function takes only the value T, or, equivalently, if, in its

truth table, the column under the statement form contains only Ts.

An example of a tautology is $(A \vee (\neg A))$, the so-called law of the excluded middle. Other simple examples are $(\neg (A \vee (\neg A)))$, $(A \Leftrightarrow (\neg(\neg A)))$, $((A \wedge B) \Rightarrow A)$, and $(A \Rightarrow (A \vee B))$

Worked out Exercise: 1. Determine whether the followings are tautologies:

a. $(((A \Rightarrow B) \Rightarrow B) \Rightarrow B)$

b. $((A \vee (\neg (B \wedge C))) \Rightarrow ((A \Rightarrow C) \vee B))$

Answer 1.a. The truth table for $(((A \Rightarrow B) \Rightarrow B) \Rightarrow B)$ is as follows:

A	B	$(A \Rightarrow B)$	$((A \Rightarrow B) \Rightarrow B)$	$(((A \Rightarrow B) \Rightarrow B) \Rightarrow B)$
T	T	T	T	T
F	T	T	T	T
T	F	F	T	F
F	F	T	F	T

As there is one F in the last column, $(((A \Rightarrow B) \Rightarrow B) \Rightarrow B)$ is not a tautology

1.b. The truth table for $((A \vee (\neg (B \wedge C))) \Rightarrow ((A \Rightarrow C) \vee B)$

The truth table is [Let $\mathfrak{B} = ((A \vee \neg (B \wedge C)))$ and $\mathcal{C} = ((A \Rightarrow C) \vee B)$]

A	B	C	$B \wedge C$	$\neg(B \wedge C)$	$\mathfrak{B}$ ~~$((A \vee (\neg(B \wedge C)))$~~	$A \Rightarrow C$	$\mathcal{C}$	$\mathfrak{B} \Rightarrow \mathcal{C}$
T	T	T	T	F	T	T	T	T
F	T	T	T	F	F	T	T	T
T	F	T	F	T	T	T	T	T
F	F	T	F	T	T	T	T	T
T	T	F	F	T	T	F	T	T
F	T	F	F	T	T	T	T	T
T	F	F	F	T	T	F	F	F
F	F	F	F	T	T	T	T	T

As there is one F in the last column, $\mathfrak{B} \Rightarrow \mathcal{C}$ is not a tautology

Exercise 2 Determine whether the followings are tautologies:

a. $(((B \Rightarrow C) \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B))$

b. $((A \Rightarrow B) \vee (B \Rightarrow A))$

c. $((\neg(A \Rightarrow B)) \Rightarrow A)$

A statement form $\mathcal{B}$ is said to logically imply a statement form $\mathcal{C}$ (or, synonymously, $\mathcal{C}$ is a logical consequence of $\mathcal{B}$) if and only if every truth assignment to the statment letters of $\mathcal{B}$ an $\mathcal{C}$ that makes $\mathcal{B}$ ~~tru~~ true also makes $\mathcal{C}$ true.

For example, $(A \wedge B)$ logically implies A, A logically implies $(A \vee B)$, and $(A \wedge (A \Rightarrow B))$ logically implies B.

Two statement forms $\mathcal{B}$ and $\mathcal{C}$ are said to be logically equivalent if $\mathcal{B}$ and $\mathcal{C}$ receive the same truth value under every assignment of the truth values of the statement letters of $\mathcal{B}$ and $\mathcal{C}$. For example, A and $(\neg(\neg A))$ are logically equivalent, as are $(A \wedge B)$ and $(B \wedge A)$

Proposition 1.1

a. $\mathcal{B}$ logically implies $\mathcal{C}$ if and only if $(\mathcal{B} \Rightarrow \mathcal{C})$ is a tautology

b. $\mathcal{B}$ and $\mathcal{C}$ are logically equivalent if and only if $(\mathcal{B} \Leftrightarrow \mathcal{C})$ is a tautology.

Proof: a (i) Assume $\mathcal{B}$ logically implies $\mathcal{C}$. Hence every truth assignment that makes $\mathcal{B}$ true also makes $\mathcal{C}$ true. Thus, no truth assignment makes $\mathcal{B}$ true and $\mathcal{C}$ false. So, no truth assignment makes $(\mathcal{B} \Rightarrow \mathcal{C})$ false, that is, every truth assignment makes $(\mathcal{B} \Rightarrow \mathcal{C})$ true. In other words, $\mathcal{B} \Rightarrow \mathcal{C}$ is a tautology.

(ii) Assume $(\mathcal{B} \Rightarrow \mathcal{C})$ is a tautology. Then for every truth assignment $(\mathcal{B} \Rightarrow \mathcal{C})$ is true, and therefore, it is not the case that $\mathcal{B}$ is true and $\mathcal{C}$ is false. Hence every assignment that makes $\mathcal{B}$ true makes $\mathcal{C}$ true. So, $\mathcal{B}$ logically implies $\mathcal{C}$.

b. $(\mathcal{B} \Leftrightarrow \mathcal{C})$ is a tautology if and only if every truth assignment makes $(\mathcal{B} \Rightarrow \mathcal{C})$ true, which is equivalent to saying that every truth assignment gives $\mathcal{B}$ and $\mathcal{C}$ the same truth value. That is, $\mathcal{B}$ and $\mathcal{C}$ are logically equivalent.

By means of a truth table, we have an effective procedure for determining whether a statement form is a tautology. Hence, by Proposition 1.1, we have effective procedure for determining whether a given statement form logically implies another given statement form and whether two given statement forms are logically equivalent.

Worked out Exercise 3. Determine whether $\mathcal{B} \Rightarrow \mathcal{C}$ is a tautology where $\mathcal{B} = ((A \Leftrightarrow ((\neg B) \vee C))$ and $\mathcal{C} = ((\neg A) \Rightarrow B)$

Ans: Here the truth table is

A	B	C	$(\neg A)$	$(\neg B)$	$((\neg B) \vee C)$	$\mathcal{B}$	$\mathcal{C}$	$\mathcal{B} \Rightarrow \mathcal{C}$
T	T	T	F	F	T	T	T	T
F	T	T	T	F	F	T	T	T
T	F	T	F	T	T	T	T	T
F	F	T	T	T	T	F	F	T
T	T	F	F	F	T	T	T	T
F	T	F	T	F	F	T	T	T
T	F	F	F	T	T	T	T	T
F	F	F	T	T	T	F	F	T

$\therefore$ $\mathcal{B} \Rightarrow \mathcal{C}$ is a tautology. (Here $\mathcal{B}$ logically implies $\mathcal{C}$)

Alternative method (without constructing truth table):

Assume that $\mathcal{B} \Rightarrow \mathcal{C}$ sometimes is F $\therefore$ $\mathcal{B}$ is T and $\mathcal{C}$ is F. Since $\mathcal{C} = ((\neg A) \Rightarrow B)$ is F, so $\neg A$ is T and B is F. Since $\neg A$ is T, A is F. Since A is F and $\mathcal{B} = ((A \Leftrightarrow ((\neg B) \vee C))$ is T. So, $(\neg B) \vee C$ is F. Since $(\neg B) \vee C$ is F, so $\neg B$ is F and C is F. Since $\neg B$ is F, B is T. But B is both T and F

Hence it is impossible that $\mathcal{B} \Rightarrow \mathcal{C}$ is false $\therefore$ $\mathcal{B} \Rightarrow \mathcal{C}$ is a tautology.

4. Determine whether the following pairs are logically equivalent.

a. $((\neg A) \vee B)$ and $((\neg B) \vee A)$

b. $(A \vee (B \Leftrightarrow C))$ and $((A \vee B) \Leftrightarrow (A \vee C))$

Ans: a. Let $\mathcal{B} = ((\neg A) \vee B)$ and $\mathcal{C} = ((\neg B) \vee A)$

The truth table for $\mathcal{B} \Leftrightarrow \mathcal{C}$ is

A	B	$\neg A$	$\neg B$	$\mathcal{B}$	$\mathcal{C}$	$\mathcal{B} \Leftrightarrow \mathcal{C}$
T	T	F	F	T	T	T
F	T	T	F	T	F	F
T	F	F	T	F	T	F
F	F	T	T	T	T	T

As there is one F in the last column, $\mathcal{B} \Leftrightarrow \mathcal{C}$ is not a tautology. So, $\mathcal{B}$ and $\mathcal{C}$ are not logically equivalent

b. Let $\mathcal{B} = (A \vee (B \Leftrightarrow C))$ and $\mathcal{C} = ((A \vee B) \Leftrightarrow (A \vee C))$

The truth table for $\mathcal{B} \Leftrightarrow \mathcal{C}$ is

A	B	C	$A \vee B$	$A \vee C$	$B \Leftrightarrow C$	$\mathcal{B}$	$\mathcal{C}$	$\mathcal{B} \Leftrightarrow \mathcal{C}$
T	T	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T	T
T	F	T	T	T	F	T	T	T
F	F	T	F	T	F	F	F	T
T	T	F	T	T	F	T	T	T
F	T	F	T	F	F	F	F	T
T	F	F	T	T	T	T	T	T
F	F	F	F	F	T	T	T	T

$\therefore$ $\mathcal{B} \Leftrightarrow \mathcal{C}$ is a tautology $\therefore$ $\mathcal{B}$ is logically equivalent to $\mathcal{C}$.

Exercise 5 Determine whether the following pairs are logically equivalent

a. $(A \Leftrightarrow B)$ and $((A \Rightarrow B) \wedge (B \Rightarrow A))$

b. $(A \wedge (B \Rightarrow C))$ and $((A \wedge B) \Leftrightarrow (A \wedge C))$