A statement form that is false for all possible truth values of its statement letters is said to be contradictory. Its truth table has only Fs in the column under the statement form. One example is $(A \Leftrightarrow (\neg A))$:

A	$\neg A$	$(A \Leftrightarrow (\neg A))$
T	F	F
F	T	F

Another is $(A \wedge (\neg A))$

~~Notice $\mathcal{B}$~~ Notice that a statement form $\mathcal{B}$ is a tautology if and only if $(\neg \mathcal{B})$ is contradictory and vice versa. ~~[If $\mathcal{B}$ is a tautology, we write $\models \mathcal{B}$]~~

A sentence that arises from a tautology by the substitution of sentences for all the statement letters, with occurrence of the same statement letter is replaced by the same sentence, is said to be logically true. An example is the English sentence, "If it is raining or it is snowing, and it is not snowing, then it is raining", which arise by substitution from the tautology $(((A \vee B) \wedge (\neg B)) \Rightarrow A)$. A statement that comes from a contradictory statement form by means of substitution is said to be logically false.

Now we prove a few general facts about tautologies.

Proposition 1.2 If $\mathcal{B}$ and $(\mathcal{B} \Rightarrow \mathcal{C})$ are tautologies then so is $\mathcal{C}$.

Proof: Assume that $\mathcal{B}$ and $(\mathcal{B} \Rightarrow \mathcal{C})$ are tautologies. If $\mathcal{C}$ takes the value F for some assignment of the truth values of the statement letters of $\mathcal{B}$ and $\mathcal{C}$, then since $\mathcal{B}$ is a tautology, $\mathcal{B}$ takes the value T and therefore, $(\mathcal{B} \Rightarrow \mathcal{C})$ takes the value F. This ~~contradict~~ contradicts the assumption that $(\mathcal{B} \Rightarrow \mathcal{C})$ is a tautology.

So, $\mathcal{C}$ is a tautology.

Proposition 1.3 If $\mathcal{J}$ be a tautology containing statement letters $A_1, A_2, \ldots, A_n$ and $\mathcal{B}$ arises from $\mathcal{J}$ by substituting statement forms $\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_n$ for $A_1, A_2, \ldots, A_n$ respectively, then $\mathcal{B}$ is a tautology.

Example: Let $\mathcal{J}$ be ~~$(A_1 A_2)$~~ $((A_1 \wedge A_2) \Rightarrow A_1)$, let $\mathcal{J}_1$ be $(B \vee C)$ and $\mathcal{J}_2$ be $(C \wedge D)$, then $\mathcal{B}$ is $(((B \vee C) \wedge (C \wedge D) \Rightarrow (B \vee C))$ and it is a tautology

Proof: Assume $\mathcal{J}$ is a tautology. For any assignment of truth values to the statement letter in $\mathcal{B}$, the forms $\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_n$ have truth values $x_1, x_2, \ldots, x_n$ (where each x_i is T or F). If we assign the values $x_1, x_2, \ldots, x_n$ to $A_1, A_2, \ldots, A_n$ respectively, then the resulting truth value of $\mathcal{J}$ is the truthvalue of $\mathcal{B}$ for the given assignment of the truth values. Since $\mathcal{J}$ is a tautology, this truth value must be T. Thus $\mathcal{B}$ always takes the value T.

Proposition 1.4 If $\mathcal{C}_1$ arises from $\mathcal{B}_1$ by substitution of $\mathcal{C}$ for one or more occurrence of $\mathcal{B}$, then $(\mathcal{B} \Leftrightarrow \mathcal{C} \Rightarrow (\mathcal{B}_1 \Leftrightarrow \mathcal{C}_1))$ is a tautology. Hence, if $\mathcal{B}$ and $\mathcal{C}$ are logically equivalent then so is $\mathcal{B}_1$ and $\mathcal{C}_1$

Example: Let $\mathcal{B}_1$ be $(\mathcal{C} \vee \mathcal{D})$, let $\mathcal{B}$ be $\mathcal{C}$ and let $\mathcal{C}$ be $(\neg(\neg\mathcal{C}))$. Then $\mathcal{C}_1$ is $((\neg(\neg\mathcal{C}) \vee \mathcal{D})$. Since $\mathcal{C}$ and $(\neg(\neg\mathcal{C}))$ are logically equivalent, $(\mathcal{C} \vee \mathcal{D})$ and $((\neg(\neg\mathcal{C}) \vee \mathcal{D})$ are also logically equivalent.

Proof: Consider any assignment of truth values to the statement letters. If $\mathcal{B}$ and $\mathcal{C}$ have opposite truth values then

then $(\mathcal{B} \Leftrightarrow \mathcal{C})$ takes the value F and hence

$((\mathcal{B} \Leftrightarrow \mathcal{C}) \Rightarrow (\mathcal{B}_1 \Leftrightarrow \mathcal{C}_1))$ takes the value T.

If $\mathcal{B}$ and $\mathcal{C}$ have the same truth value then so do $\mathcal{B}_1$ and $\mathcal{C}_1$ since $\mathcal{C}_1$ differs from $\mathcal{B}_1$ only in containing $\mathcal{C}$ in some places where $\mathcal{B}_1$ contains $\mathcal{B}$. Therefore in this case $(\mathcal{B} \Leftrightarrow \mathcal{C})$ is T and $(\mathcal{B}_1 \Leftrightarrow \mathcal{C}_1)$ is T, and so $((\mathcal{B} \Leftrightarrow \mathcal{C}) \Rightarrow (\mathcal{B}_1 \Leftrightarrow \mathcal{C}_1))$ is T

6. Apply proposition 1.3 when $\mathcal{J}$ is $A_1 \Rightarrow A_1 \vee A_2$, $\mathcal{J}_1$ is $B \wedge D$ and $\mathcal{J}_2$ is $\neg B$.

Ans: $\mathcal{B}$, arising from $\mathcal{J}$ by substituting $\mathcal{J}_1$ and $\mathcal{J}_2$ for A_1 and A_2, is $((B \wedge D) \Rightarrow (B \wedge D) \vee (\neg B)$. Here $\mathcal{J}$ is a tautology, as the truth table below shows.

A_1	A_2	$A_1 \vee A_2$	$A_1 \Rightarrow A_1 \vee A_2$
T	T	T	T
F	T	T	T
T	F	T	T
F	F	F	T

So, by proposition 1.3 $\mathcal{B}$ is a tautology.

7. Apply Proposition 1.4 when $\mathcal{B}_1$ is $(B \Rightarrow C) \wedge D$, $\mathcal{B}$ is $B \Rightarrow C$ and $\mathcal{C}$ is $\neg B \vee C$

Ans: Substituting $\mathcal{C}$ for $\mathcal{B}$ in $\mathcal{B}_1$, we have $\mathcal{C}_1$ which is $(\neg B \vee C) \wedge D$. Then ~~applying Proposition 1.4, we have~~ ~~$((B \Rightarrow C) \Rightarrow \neg B \vee C)$~~ Here $\mathcal{B}$ and $\mathcal{C}$ are logically equivalent as shown in the following truth table

B	$\neg B$	C	$\mathcal{B}$	$\mathcal{C}$	$\mathcal{B} \Rightarrow \mathcal{C}$
T	F	T	T	T	T
F	T	T	T	T	T
T	F	F	F	F	T
F	T	F	T	T	T

So, by Proposition 1.4 $\mathcal{B}_1$ is logically equivalent to $\mathcal{C}_1$.

Exercise 7. Show that each statement form in column I is logically equivalent to the form next to it in column II. Write $\mathcal{B} \equiv \mathcal{C}$ if $\mathcal{B}$ and $\mathcal{C}$ are logically equivalent.

I		II	
a. $A \wedge T$	$\equiv$	A	(Identity laws) T means Truth, F means False
b. $A \vee F$	$\equiv$	A	
c. $A \vee T$	$\equiv$	T	(Domination laws)
d. $A \wedge F$	$\equiv$	F	
e. $A \wedge A$	$\equiv$	A	(Idempotent Laws)
f. $A \vee A$	$\equiv$	A	

g. $A \Rightarrow (B \Rightarrow C) \equiv (A \wedge B) \Rightarrow C$

h. $(A \wedge B) \vee \neg B \equiv A \vee \neg B$

i. $(A \vee B) \wedge \neg B \equiv A \wedge \neg B$

j. $\neg(\neg A) \equiv A$ (Double negation law)

k. $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ (Distributive Laws)

l. $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$

m. $A \Rightarrow B \equiv \neg B \Rightarrow \neg A$ (Law of Contrapositive)

n. $A \Leftrightarrow B \equiv B \Leftrightarrow A$ (Biconditional Commutativity)

o. $(A \Leftrightarrow B) \Leftrightarrow C \equiv A \Leftrightarrow (B \Leftrightarrow C)$ (Biconditional Associativity)

p. (i) $A \wedge B \equiv B \wedge A$ (Commutative Laws)
(ii) $A \vee B \equiv B \vee A$

q. $A \Leftrightarrow B \equiv (A \wedge B) \vee (\neg A \wedge \neg B)$

r. $\neg(A \Rightarrow B) \equiv A \Leftrightarrow \neg B$

s. (i) $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$ Associative laws
(ii) $(A \vee B) \vee C \equiv A \vee (B \vee C)$

t. (i) $A \vee (A \wedge B) \equiv A$ (Absorption Laws)
(ii) $A \wedge (A \vee B) \equiv A$

u. (i) $\neg(A \vee B) \equiv (\neg A) \wedge (\neg B)$ De Morgan's Laws
(ii) $\neg(A \wedge B) \equiv (\neg A) \vee (\neg B)$

v. (i) $A \vee \neg A \equiv T$ Negation laws.
(ii) $A \wedge \neg A \equiv F$

w. $A \Rightarrow B \equiv \neg A \vee B$

8. Without using truth table prove that $\neg(A \vee B) \vee (\neg A \wedge B) \equiv \neg A$

Solution: $\neg(A \vee B) \vee (\neg A \wedge B) \equiv (\neg A \wedge \neg B) \vee (\neg A \wedge B)$ [De Morgan's Law]

$\equiv \neg A \wedge (\neg B \vee B)$

$\equiv \neg A \wedge T$ [$\neg B \vee B \equiv T$]

$\equiv \neg A$

Hence $\neg(A \vee B) \vee (\neg A \wedge B) \equiv \neg A$

9. Without using truth table prove that

$A \Rightarrow (B \Rightarrow C) \equiv (A \wedge B) \Rightarrow C$

Solution: $A \Rightarrow (B \Rightarrow C) \equiv A \Rightarrow (\neg B \vee C)$ [As $A \Rightarrow B \equiv \neg A \vee B$]

$\equiv \neg A \vee (\neg B \vee C)$ [Using Same result]

$\equiv (\neg A \vee \neg B) \vee C$ [Associative law]

$\equiv \neg(A \wedge B) \vee C$ [De Morgan's Law]

$\equiv A \wedge B \Rightarrow C$ [As $\neg A \vee B \equiv A \Rightarrow B$]

Hence $A \Rightarrow (B \Rightarrow C) \equiv (A \wedge B) \Rightarrow C$

Exercise 10. Without using truth table, prove the followings :

(i) $\neg A \wedge (\neg B \wedge C) \vee (B \wedge C) \vee (A \wedge C) \equiv C$

(ii) $A \Leftrightarrow B \equiv (A \vee B) \rightarrow (A \wedge B)$

(iii) $(A \wedge (\neg A \vee B)) \vee (B \wedge \neg(A \wedge B)) \equiv B$

(iv) $A \Rightarrow (B \Rightarrow C) \equiv A \Rightarrow (\neg B \vee C) \equiv (A \wedge B) \rightarrow C$

(v) $A \Rightarrow (B \vee C) \equiv (A \Rightarrow B) \vee (A \Rightarrow C)$

11. Show that the following statement forms are contradictory :

(i) $(A \vee B) \wedge (\neg A \wedge \neg B)$

(ii) $A \wedge (B \wedge \neg A)$

(iii) $(\neg A \wedge B) \wedge (\neg B \wedge (A \Rightarrow B))$

Proof (i) The truth table for $(A \vee B) \wedge (\neg A \wedge \neg B)$ is

A	B	$\neg A$	$\neg B$	$A \vee B$	$\neg A \wedge \neg B$	$(A \vee B) \wedge (\neg A \wedge B)$
T	T	F	F	T	F	F
F	T	T	F	T	F	F
T	F	F	T	T	F	F
F	F	T	T	F	~~F~~T	F

So, from the last column of the truth table, we see that
$(A \vee B) \wedge (\neg A \wedge \neg B)$ is contradictory

(ii) Exercise

(iii) Exercise

~~Then~~ we know that $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ ——(1)

~~In Changing V~~ Interchanging $\vee$ and $\wedge$, we have

$$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C) \quad \text{——(2)}$$

(1) and (2) are said to be dual of each other.

Adequate sets of connectives

Every statement form containing n statement letters generates a corresponding truth function of n arguments, denoted by $f(x_1, x_2, \dots x_n)$ whose domain ~~is the n-tuples of T and F~~ is $\{(x_1, x_2, \dots, x_n) : x_i = T \text{ or } F, i=1, \dots n\}$ and $f(x_1, \dots x_n) = T$ or F. Logically equivalent form generate the same truth functions. A natural question ~~arises~~ is whether all truth functions are so generated.

We state a related theorem without proof.

Proposition 1.5 Every truth function is generated by a statement form involving the connectives $\neg$, $\wedge$ and $\vee$

Example 1.

x_1	x_2	$f(x_1, x_2)$
T	T	F
F	T	T
T	F	T
F	F	T

Here the statement form $\mathcal{B}$ generated by the truth function

D given by $(\neg A_1 \wedge A_2) \vee (A_1 \wedge \neg A_2) \vee (\neg A_1 \wedge \neg A_2)$

[For those rows in the truth where $f(x_1, x_2)$ is T, ~~we form~~ i.e., for row 2, row 3 and row 4, we form ~~$\mathcal{B}$~~, statement forms C_2, C_4 & C_5 such that $C_2 = (\neg A_1 \wedge A_2)$ (as x_1 is F and x_2 is T in row 2), $C_4 = (A_1 \wedge \neg A_2)$, $C_5 = (\neg A_1 \wedge \neg A_2)$ (as in the same way formed as C_2] So, $D = (\neg A_1 \wedge A_2) \vee (A_1 \wedge \neg A_2) \vee (\neg A_1 \wedge \neg A_2)$

[This is the way to generate a proof of Proposition 1.5]

Ex.2

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
T	T	T	T
F	T	T	F
T	F	T	T
F	F	T	T
T	T	F	F
F	T	F	F
T	F	F	F
F	F	F	T

Here the statement form D is given by

$$(A_1 \wedge A_2 \wedge A_3) \vee (A_1 \wedge \neg A_2 \wedge A_3) \vee (\neg A_1 \wedge \neg A_2 \wedge A_3) \vee (\neg A_1 \wedge \neg A_2 \wedge \neg A_3)$$

Corollary 1.6 Every truth function can be generated by a statement form containing as connectives only $\wedge$ and $\neg$, or only $\vee$ and $\neg$, or only $\Rightarrow$ and $\neg$

Proof: Notice that $\mathcal{B} \vee \mathcal{C}$ is logically equivalent to $\neg(\neg \mathcal{B} \wedge \neg \mathcal{C})$. Hence, by second part of Proposition 1.4, any statement in $\wedge$, $\vee$ and $\neg$ is logically equivalent to a statement form in only $\wedge$ and $\neg$ (obtained by replacing all expressions $\mathcal{B} \vee \mathcal{C}$ by $\neg(\neg \mathcal{B} \wedge \neg \mathcal{C})$. The other parts of the corollary are similar consequences of the following tautologies :

$\mathcal{B} \wedge \mathcal{C} \Leftrightarrow \neg(\neg\mathcal{B} \vee \neg\mathcal{C})$, $\mathcal{B} \vee \mathcal{C} \Leftrightarrow (\neg\mathcal{B} \Rightarrow \mathcal{C})$ and $\mathcal{B} \wedge \mathcal{C} \Leftrightarrow \neg(\mathcal{B} \Rightarrow \neg\mathcal{C})$

Conjunctive and disjunctive normal forms : By a literal we mean a statement letter or a negation of a statement letter. A statement form is said to be disjunctive normal form (dnf) if it is a disjunction consisting one or more disjuncts, each of which is a conjunction of one or more literals - For example,

$(A \wedge B) \vee (\neg A \wedge C)$, $(A \wedge B \wedge \neg A) \vee (C \wedge \neg B) \vee (A \wedge \neg C)$, A, $A \wedge B$ and $A \vee (B \wedge C)$

A form is in conjunctive normal form (cnf) if it is a conjunction of one or more conjuncts, each of which is a disjunction of one or more literals - For examples, $(B \vee C) \wedge (A \vee B)$, $(B \vee \neg C) \wedge (A \vee D)$, $A \wedge (B \vee A) \wedge (\neg B \vee A)$, $A \vee \neg B$, $A \wedge B$, A

From the proof of Proposition 1.5 shows every statement form $\mathcal{B}$ is logically equivalent to one in disjunctive normal forms. By applying the result to $\neg\mathcal{B}$, prove that $\mathcal{B}$ is also logically equivalent to a form in conjunctive normal form.

3. Find logically equivalent dnfs and cnfs for
(i) $\neg(A \Rightarrow B) \vee (\neg A \wedge C)$ and (ii) $A \Leftrightarrow ((B \wedge \neg A) \vee C)$

Solution (i) Let $\mathcal{B} = \neg(A \Rightarrow B) \vee (\neg A \wedge C)$. The truth table for $\mathcal{B}$ and $\neg\mathcal{B}$ is

A	B	C	$\neg A$	$A \Rightarrow B$	$\neg(A \Rightarrow B)$	$\neg A \wedge C$	$\mathcal{B}$	$\neg\mathcal{B}$
T	T	T	F	T	F	F	F	T
F	T	T	T	T	F	T	T	F
T	F	T	F	F	T	F	T	F
F	F	T	T	T	F	T	T	F
T	T	F	F	T	F	F	F	T
F	T	F	T	T	F	F	F	T
T	F	F	F	F	T	F	T	F
F	F	F	T	T	F	F	F	T

$\therefore \mathcal{B}$ is logically equivalent to the dnf

$(\neg A \wedge B \wedge C) \vee (A \wedge \neg B \wedge C) \vee (\neg A \wedge \neg B \wedge C) \vee (A \wedge \neg B \wedge \neg C)$

and $\neg\mathcal{B}$ is logically equivalent to the dnf

$(A \wedge B \wedge C) \vee (A \wedge B \wedge \neg C) \vee (\neg A \wedge B \wedge \neg C) \vee (\neg A \wedge \neg B \wedge \neg C)$

So, $\mathcal{B}$ is logically equivalent to the cnf (by De Morgan's law)

$(\neg A \vee \neg B \vee \neg C) \wedge (\neg A \wedge \neg B \wedge C) \wedge (A \vee \neg B \vee C) \wedge (A \wedge B \wedge C)$

(ii) Let $\mathcal{B} = A \Leftrightarrow ((B \wedge \neg A) \vee C)$. The truth table for $\mathcal{B}$ and $\neg\mathcal{B}$ is

A	B	C	$\neg A$	$B \wedge \neg A$	$(B \wedge \neg A) \vee C$	$\mathcal{B}$	$\neg\mathcal{B}$
T	T	T	F	F	T	T	F
F	T	T	T	T	T	F	T
T	F	T	F	F	T	T	F
F	F	T	T	F	T	F	T
T	T	F	F	F	F	F	T
F	T	F	T	T	T	F	T
T	F	F	F	F	F	F	T
F	F	F	T	F	F	T	F

So, $\mathcal{B}$ is logically equivalent to the dnf

$(A \wedge B \wedge C) \vee (A \wedge \neg B \wedge C) \vee (\neg A \wedge \neg B \wedge \neg C)$

and $\neg\mathcal{B}$ is logically equivalent to the dnf

$(\neg A \wedge B \wedge C) \vee (\neg A \wedge \neg B \wedge C) \vee (A \wedge B \wedge \neg C) \vee (\neg A \wedge B \wedge \neg C) \vee (A \wedge \neg B \wedge \neg C)$

So, $\mathcal{B}$ is logically equivalent to the cnf (by De Morgan's law)

$(A \vee \neg B \vee \neg C) \wedge (A \vee B \vee \neg C) \wedge (\neg \vee \neg B \vee C) \wedge (A \vee \neg B \vee C) \wedge (\neg A \vee B \vee C)$

Ex.4 For each of the following, find a logically equivalent dnf and cnf

(i) $(A \vee B) \wedge (\neg B \vee C)$

(ii) $\neg A \vee (B \Rightarrow \neg C)$

(iii) $(A \wedge \neg B) \vee (A \wedge C)$

(iv) $(A \vee B) \Leftrightarrow \neg C$

A dnf (cnf) is called full if no disjunct (conjunct) contains two occurences of literals with the same letter and if a letter that occurs in one disjunct (conjunct) also occurs in all others.

For example, $(A \wedge \neg A \wedge B) \vee (A \wedge B)$, $(B \wedge B \wedge C) \vee (B \wedge C)$ and $(B \wedge C) \vee B$ are not full, whereas $(A \wedge B \wedge \neg C) \vee (A \wedge B \wedge C) \vee (A \wedge \neg B \wedge \neg C)$ and $(A \wedge \neg B) \vee (B \wedge A)$ are full dnfs.

5. Find full dnfs and cnfs logically equivalent to

(i) $(A \wedge B) \vee \neg A$ and (ii) $\neg(A \Rightarrow B) \vee (\neg A \wedge C)$

Solution: (i) Let $\mathscr{B} = (A \wedge B) \vee \neg A$. Truth table for $\mathscr{B}$ and $\neg \mathscr{B}$ is

A	B	$\neg A$	$A \wedge B$	$\mathscr{B}$	$\neg \mathscr{B}$
T	T	F	T	T	F
F	T	T	F	T	F
T	F	F	F	F	T
F	F	T	F	T	F

$\therefore$ $\mathscr{B}$ is logically equivalent to full dnf

$(A \wedge B \wedge \neg C) \vee (\neg A \wedge B \wedge C) \vee (\neg A \wedge \neg B \wedge C)$

$\neg \mathscr{B}$ is logically equivalent to full dnf

$(A \wedge \neg B \wedge \neg C)$

So $\mathscr{B}$ is logically equivalent to full cnf (by De Morgan's Law)

$(\neg A \vee B \vee C)$.

(ii) Do it yourself.

Ex. 6 Find the logically equivalent full dnf and cnf for all statement forms in Ex. 4.

Ex. 7 construct statement forms in $\neg$ and $\wedge$ (respectively, in $\neg$ and $\vee$ or in $\neg$ and $\Rightarrow$) logically equivalent to the statement forms in Ex. 4.

i) $(A \vee B) \wedge (\neg B \vee C) \equiv \neg(\neg A \wedge \neg B) \wedge \neg(B \wedge \neg C)$. Do the other cases using corollary 1.6

(ii), (iii) and (iv) Do it yourself