c. Not all birds can fly

d. All birds can not fly

e. x is transcendental only if it is irrational

f. Seniors advise only juniors

g. If anyone can solve the problem, Hirak can.

h. Nobody loves a loser

i. Nobody in the statistics class is smarter than everyone in the logic class

j. Vivek hates all people who do not hate themselves.

k. Everyone hates somebody and no one hates everybody, or somebody hates everybody and someone hates nobody.

l. You can fool some of the people all of the time and you can fool all the people some of the time, but you can't ~~fo~~ fool all the people all the time.

m. Any sets that have the same members are equal.

n. Anyone who knows Anawar likes him.

o. There is no set belonging to precisely those sets that do not belong to themselves

p. There is no barber who shaves precisely those ~~sets~~ men who do not shave themselves.

<u>Exercise 2.7</u> Translate the following into everyday English

a. $(\forall x)(M(x) \wedge [(\forall y)\neg W(x,y) \Rightarrow U(x))$, where $M(x)$ means x is a man, $W(x,y)$ means x is a friend of y and $U(x)$ means x is unhappy.

b. $(\forall x)(V(x) \wedge P(x) \Rightarrow A(x,b))$, where $V(x)$ means x is an even integer, $P(x)$ means x is a prime integer, $A(x,y)$ means $x=y$, and ~~two~~ b denotes 2.

c. $\neg(\exists y)(I(y) \wedge (\forall x)(I(x) \Rightarrow L(x,y)))$, where $I(y)$ means y is an integer and $L(x,y)$ means $x \leq y$

d. In the following wfs, $A_1^1(x)$ means x is a person and $A_1^2(x,y)$ means x hates y.

(i) $(\exists x)(A_1^1(x) \wedge (\forall y)(A_1^1(y) \Rightarrow A_1^2(x,y)))$

(ii) $(\forall x)(A_1^1(x) \Rightarrow (\forall y)(A_1^1(y) \Rightarrow A_1^2(x,y)))$

(iii) $(\exists x)(A_1^1(x) \wedge (\forall y)(A_1^1(y) \Rightarrow (A_1^2(x,y) \Leftrightarrow A_1^2(y,y))))$

e. $(\forall x)(H(x) \Rightarrow (\exists y)(\exists z)(\neg A(y,z) \wedge (\forall u)(P(u,x) \Leftrightarrow (A(u,y) \vee A(u,z)))))$, where $H(x)$ means x is a person, $A(u,v)$ means "$u=v$", and $P(u,x)$ means u is a parent of x.

3. First Order Languages and their Interpretations, Satisfiability and Truth and Models.

well formed formulas have meaning only when an interpretation is given for the symbols. We usually are interested in interpreting wfs whose symbols come from a specific Language. For this reason, we shall define the notion of a first order language.

Definition : A first order language $\mathcal{L}$ contains the following symbols:

a. The propositional connectives $\neg$ and $\Rightarrow$, and the universal quantifier symbol $\forall$.

b. Punctuation marks: the left Parenthesis "(", the right parenthesis ")", and the Comma ",".

c. Denumerably many individual variables $x_1, x_2, \ldots$

d. A finite or denumerable, possibly empty, set of function letters

e. A finite or denumerable, possibly empty, set of individual constants.

f. A non-empty set of predicate letters

By a term of $\mathcal{L}$ we mean a term whose symbols are symbols of $\mathcal{L}$. By a wf of $\mathcal{L}$ we mean a wf whose symbols are symbols of $\mathcal{L}$. Thus, in a language $\mathcal{L}$, some or all of the function letters and individual constants may be absent, and some (but not all) of the predicate letters may be absent (If there were no predicate letters, there would be no wfs). The individual constants, function letters and predicate letters of a language $\mathcal{L}$ are called non-logical constants of $\mathcal{L}$. Languages are designed in ~~accordan~~ accordance with the subject matter we wish to study. A language for arithmetic might contain function letters for addition and multiplication and a predicate letter for equality, whereas a language for geometry is likely to have predicate letters for equality and the notions of point and line, but no function letters at all.

Definition: Let $\mathcal{L}$ be a first order language. An interpretation M of $\mathcal{L}$ consists of the following ingrediants:

a. A non-empty set D, called the domain of the interpretation.

b. For each predicate letter A_j^n of $\mathcal{L}$, an assignment of an n-place relation $(A_j^n)^M$ in D (i.e., a subset of D^n)

c. For each function letter f_j^n of $\mathcal{L}$, an assignment of an n-place operation $(f_j^n)^M$ in D (i.e., a function from D^n into D).

d. For each individual constant a_i of $\mathcal{L}$, an assignment of some fixed element $(a_i)^M$ of D.

Given such an interpretation, variables are thought of as ranging over the set D, and $\neg$, $\Rightarrow$ and quantifiers are given their usual meaning.

For a given interpretation of a language $\mathcal{L}$, a wf of $\mathcal{L}$ without free variables (called a closed wf or a sentence) represents a proposition that is true or false, whereas a wf with free variables may be satisfied (i.e., true) for some values in the domain and not satisfied (i.e., false) for the others.

Examples consider the following wfs:

1. $A_1^2(x_1, x_2)$
2. $(\forall x_2)\, A_1^2(x_1, x_2)$
3. $(\exists x_1)(\forall x_2)\, A_1^2(x_1, x_2)$

Let us take as domain the set of all positive integers and interpret $A_1^2(y,z)$ as $y \le z$. Then wf 1 represents the expression "$x_1 \le x_2$" which is satisfied by all pair (a,b) of positive integers such that $a \le b$. Wf 2 represents the expression "For all positive integers x_2, $x_1 \le x_2$," which is satisfied only by the integer 1. Wf 3 is a true sentence ~~assert~~ asserting that there is a smallest positive integer. If we ~~were~~ were to take as the domain the set of all integers, then wf 3 would be false.

Exercise 2.8 For the following wfs and for the given interpretations, indicate for what values the wfs are satisfied (if they contain free variables) or whether they are true or false (if they are closed wfs).

i. $A_1^2(f_1^2(x_1, x_2), a_1)$

ii. $A_1^2(x_1, x_2) \Rightarrow A_1^2(x_2, x_1)$

iii. $(\forall x_1)(\forall x_2)(\forall x_3)\left(A_1^2(x_1, x_2) \wedge A_1^2(x_2, x_3) \Rightarrow A_1^2(x_1, x_3)\right)$

a. The domain is the set of positive integers, $A_1^2(y,z)$ is $y \ge z$,

$f_1^2(y,z)$ is $y.z$ and a_1 is 2.

b. The domain is the set of integers, $A_1^2(y,z)$ is $y=z$, $f_1^2(y,z)$ is $y+z$ and a_1 is 0.

c. The domain is the set of all sets of integers, $A_1^2(y,z)$ is $y \subseteq z$, $f_1^2(y,z)$ is $y \cap z$, and a_1 is the empty set ϕ.

Exercise 2.9 Describe in Everyday English the assertions determined by the following wfs and interpretation

a. $(\forall x)(\forall y)(A_1^2(x,y) \Rightarrow (\exists z)(A_1^1(z) \wedge A_1^2(x,z) \wedge A_1^2(z,y)))$, where the domain D is the set of real numbers, $A_1^2(x,y)$ means $x<y$, and $A_1^1(z)$ means z is a rational number.

b. $(\forall x)(A_1^1(x) \Rightarrow (\exists y)(A_2^1(y) \wedge A_1^2(y,x)))$, where D is the set of all days and people, $A_1^1(x)$ means x is a day, $A_2^1(y)$ means y is a footballer, and $A_1^2(y,x)$ means y is born on day x.

c. $(\forall x)(\forall y)(A_1^1(x) \wedge A_1^1(y) \Rightarrow A_2^1(f_1^2(x,y)))$, where D is the set of integers, $A_1^1(x)$ means x is odd, $A_2^1(x)$ means x is even, $f_1^2(x,y)$ denotes $x+y$.

d. For the following wfs, D is the set of all people and $A_1^2(u,v)$ means u hates v.

i. $(\exists x)(\forall y)A_1^2(x,y)$

ii. $(\forall y)(\exists x)A_1^2(x,y)$

iii. $(\exists x)(\forall y)(\forall z)(A_1^2(y,z) \Rightarrow A_1^2(x,y))$

iv. $(\exists x)(\forall y)\neg A_1^2(x,y)$

e. $(\forall x)(\forall u)(\forall v)(\forall w)(E(f(u,u),x) \wedge E(f(v,v),x) \wedge E(f(w,w),x) \Rightarrow E(u,v) \vee E(u,w) \vee E(v,w))$, where D is the set of real numbers, $E(x,y)$ means $x=y$ and f denotes the multiplication operation.

f. $A_1^1(x_1) \wedge (\exists x_3)(A_2^2(x_1,x_3) \wedge A_2^2(x_3,x_2))$ where D is the set of people, $A_1^1(u)$ means u is a woman and $A_2^2(u,v)$ means u is a parent of v.

g. $(\forall x_1)(\forall x_2)(A_1^1(x_1) \wedge A_1^1(x_2) \Rightarrow A_2^1(f_1^2(x_1,x_2)))$ where D is the set of real numbers, $A_1^1(u)$ means u is negative, $A_2^1(u)$ means u is positive, and $f_1^2(u,v)$ is the product of u and v.

We will now define satisfiability, on the basis of which the notion of truth will be defined. Moreover, instead of talking about the n-tuples of of objects that satisfy a wf that has n free variables, it is much more convenient from a technical standpoint to deal uniformly with denumerable sequences. What we have in mind is that a denumerable sequence $s = (s_1, s_2, s_3, \ldots)$ is to be thought of as satisfying a wf $\mathcal{B}$ that $x_{j_1}, x_{j_2}, \ldots, x_{j_n}$ as free variables (where $j_1 < j_2 < \cdots < j_n$) if the n-tuple $(s_{j_1}, s_{j_2}, \ldots, s_{j_n})$ satisfies $\mathcal{B}$ in the usual sense. For example, a denumerable sequence $(s_1, s_2, s_3, \ldots)$ of objects in the domain of an interpretation M will turn out to satisfy the wf $A_1^2(x_2, x_5)$ if and only if the ordered pair, (s_2, s_5) is in the relation $(A_1^2)^M$ assigned to the predicate letter A_1^2 by the interpretation.

Let M be an interpretation of a language $\mathcal{L}$ and let D be the domain of M. Let Σ be the set of all denumerable sequences of elements of D. For a wf $\mathcal{B}$ of $\mathcal{L}$, we shall define what it means for a sequence $s = (s_1, s_2, \ldots)$ in Σ to satisfy $\mathcal{B}$ in M. As a preliminary step, for a given s in Σ we shall define a function s^* that assigns to each term t of $\mathcal{L}$ an element $s^*(t)$ in D.

1. If t is a variable x_j, let $s^*(t)$ be s_j
2. If t is an individual constant a_j, then $s^*(t)$ is the interpretation $(a_j)^M$ of this constant.
3. If f_k^n is a function letter, $(f_k^n)^M$ is the corresponding operation in D, and $t_1, t_2, \ldots, t_n$ are terms, then

$$s^*\left(f_k^n(t_1, t_2, \ldots, t_n)\right) = (f_k^n)^M\left(s^*(t_1), s^*(t_2), \ldots, s^*(t_n)\right).$$

Intuitively, $s^*(t)$ is the element of D obtained by substituting, for each j, a name s_j for all occurrences of x_j in t and then performing the operations of the interpretation corresponding to the function letters of t. For instance, if t is $f_2^2(x_3, f_1^2(x_1, a_1))$ and if the interpretation has the set of integers as its domain f_2^2 and f_1^2 are interpreted as ordinary multiplication and addition, respectively and a_1 is interpreted as 2, then for any sequence $s = (s_1, s_2, \cdots)$ of integers, $s^*(t)$ is the integer $s_3 \cdot (s_1 + 2)$. This is really nothing more than the ordinary way of reading mathematical expressions.

Now we proceed to the definition of satisfaction, which will be an inductive definition.

1. If $\mathcal{B}$ is atomic wf $A_k^n(t_1, \cdots, t_n)$ and $(A_k^n)^M$ is the corresponding n-place relation of the interpretation, then a sequence $s = (s_1, s_2, \cdots)$ satisfies $\mathcal{B}$ if and only if $\cancel{(A_k^n)^M(s_0^*(t_1), \cdots, s^*(t}$ $(A_k^n)^M(s^*(t_1), \cdots s^*(t_n))$ – that is, if n-tuples $(s^*(t_1), \cdots, s^*(t_n))$ is in the relation $(A_k^n)^M$.
[For example, if the domain of interpretation is the set of real numbers, the interpretation of A_1^2 is the relation $\leq$, and the interpretation of f_1^1 is the function e^x, then a sequence $s = (s_1, s_2, \cdots)$ of real numbers satisfies $A_1^2(f_1^1(x_2), x_5)$ if and only if $e^{s_2} \leq s_5$. If the domain is the set of integers, the interpretation of $A_1^4(x, y, u, v)$ is $x \cdot v = u \cdot y$ and the interpretation of a_1 is 3, then a sequence $s = (s_1, s_2, \cdots)$ of integers satisfies $A_1^4(x_3, a_1, x_1, x_3)$ if and only if $s_3^2 = 3s_1$]

2. s satisfies $\neg\mathcal{B}$ if and only if s does not satisfy $\mathcal{B}$.
3. s satisfies $\mathcal{B} \Rightarrow \mathcal{C}$ if and only if s does not satisfy $\mathcal{B}$ or s satisfies $\mathcal{C}$.
4. s satisfies $(\forall x_i)\mathcal{B}$ if and only if every sequence that differs from s in at most ~~ith~~ the ith component satisfies $\mathcal{B}$. [In other words, a sequence $s = (s_1, s_2, \cdots, s_i, \cdots)$ satisfies $(\forall x_i)\mathcal{B}$ if and only if, for every element c of the domain, the sequence $(s_1, s_2, \cdots, c, \cdots)$ satisfies $\mathcal{B}$. Here $(s_1, s_2, \cdots, c, \cdots)$ denotes the sequence obtained from $(s_1, s_2, \cdots, s_i, \cdots)$ by replacing the ith component s_i by c. Note also that if s satisfies $(\forall x_i)\mathcal{B}$, then as a special case, s satisfies $\mathcal{B}$.]