For Fig:7 and Fig:8, do it yourself.

A statement form is satisfiable if it is true for some assignment of truth values of its statement letter.

Ex 10. Show whether $A \vee (C \wedge \neg B)$ is satisfiable or not.

The truth table of $A \vee (C \wedge \neg B)$ is

A	B	C	$\neg B$	$C \wedge \neg B$	$A \vee (C \wedge \neg B)$
T	T	T	F	F	T
F	T	T	F	F	F
T	F	T	T	T	T
F	F	T	T	T	T
T	T	F	F	F	T
F	T	F	F	F	F
T	F	F	T	F	T
F	F	F	T	F	F

As in the last column there is at least one T, so $A \vee (C \wedge \neg B)$ is satisfiable

Ex 11 Determine whether the following statement forms are satisfiable or not:

(i) $(A \vee B) \wedge (\neg A \vee B \vee C) \wedge (\neg A \vee \neg B \vee \neg C)$

(ii) $((A \Rightarrow B) \vee C) \Leftrightarrow (\neg B \wedge (A \wedge C))$

Solution: Do it yourself.

Any statement form $\mathcal{B}$ is satisfiable if and only if $\neg \mathcal{B}$ is not a tautology.

An Axiom System for the Propositional Calculus

Truth tables enables us to answer many of the significant questions concerning the truth functional connectives, such as whether a given statement form is a tautology, contradictory or neither of them, whether it logically implies or logically equivalent to some other given statement form. The more complex parts of logic we shall treat later can not be handled by truth tables or by any other similar effective procedure. Consequently, another approach, that we will try, is formal axiomatic theory.

A formal theory $\mathcal{S}$ is defined when the following conditions are satisfied:

1. A countable set of symbols is given as the symbols of $\mathcal{S}$. A finite sequence of symbols of $\mathcal{S}$ is called an expression of $\mathcal{S}$.
2. There is a subset of the set of expressions of $\mathcal{S}$ called the set of well-formed formulas (wfs) of $\mathcal{S}$. There is usually an effective procedure to determine whether a given expression is a wf.
3. There is a set of wfs called the set of axioms of $\mathcal{S}$. Most often, one can effectively decide whether a given wf is an axiom; in such a case, $\mathcal{S}$ is called an axiomatic theory.
4. There is a finite set $R_1, R_2, \ldots, R_n$ of relations among wfs, called rules of inference. For each R_i, there is a unique positive integer j such that, for every set of j wfs and each wf $\mathcal{B}$, one can effectively decide whether the given j wfs are in the relation R_i to $\mathcal{B}$, and, if so, $\mathcal{B}$ is said to follow from or to be a direct consequence of the given wfs by virtue of R_i. [An example of a rule of inference will be the rule modus ponens (MP): $\mathcal{C}$ follows from $\mathcal{B}$ and $\mathcal{B} \Rightarrow \mathcal{C}$. According to our precise definition, this rule is the relation consisting of all ordered triples $(\mathcal{B}, \mathcal{B} \Rightarrow \mathcal{C}, \mathcal{C})$ where $\mathcal{B}$ and $\mathcal{C}$ are arbitrary wfs of the formal system.]

A proof in $\mathcal{S}$ is a sequence $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$ of wfs such that, for each i, either $\mathcal{B}_i$ is an axiom of $\mathcal{S}$ or $\mathcal{B}_i$ is a direct consequence of some of the preceeding wfs in the sequence by virtue of one of the rules of inference of $\mathcal{S}$.

A theorem of $\mathcal{S}$ is a wf $\mathcal{B}$ of $\mathcal{S}$ such that $\mathcal{B}$ is the last wf of some proof in $\mathcal{S}$. Such a proof is called a proof of $\mathcal{B}$ in $\mathcal{S}$.

Even if $\mathcal{S}$ is axiomatic – that is, if there is an effective procedure for checking any given wf to see whether it is an axiom – the notion of "theorem" is not necessarily effective since, in general, there is no effective procedure for determining, given any wf $\mathcal{B}$, whether there is a proof of $\mathcal{B}$. A theory for which there is such an effective procedure is said to be <u>decidable</u>, otherwise, the theory is said to be <u>undecidable</u>. From an intuitive standpoint, a decidable theory is one for which a machine can be devised to test wfs for theoremhood, whereas, for an undecidable theory, ingenuity is required to determine whether wfs are theorem.

A wf $\mathcal{C}$ is said to be a consequence in $\mathcal{S}$ of a set Γ of wfs if and only if there is a sequence $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k$ of wfs such that $\mathcal{C}$ is $\mathcal{B}_k$ and, for each i, either $\mathcal{B}_i$ is an axiom or $\mathcal{B}_i$ is in Γ or $\mathcal{B}_i$ is a direct consequence by some rule of inference of ~~the~~ some of the preceding wfs in the sequence. Such a sequence is called a proof (or deduction) of $\mathcal{C}$ from Γ. The members of Γ are called the hypotheses or premisses of the proof. We use $\Gamma \vdash \mathcal{C}$ as an abbreviation for "$\mathcal{C}$ is a consequence of Γ".

If Γ is a finite set $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m\}$ we write $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m \vdash \mathcal{C}$ instead of $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m\} \vdash \mathcal{C}$. If Γ is the empty set ϕ, then $\phi \vdash \mathcal{C}$ if and only if $\mathcal{C}$ is a theorem. It is customary to omit the sign "ϕ" and simply write $\vdash \mathcal{C}$. Thus $\vdash \mathcal{C}$ is another way of asserting that $\mathcal{C}$ is a theorem.

The following are simple properties of the notion of consequence:

1. If $\Gamma \subseteq \Delta$ and $\Gamma \vdash \mathcal{C}$, then $\Delta \vdash \mathcal{C}$.
2. $\Gamma \vdash \mathcal{C}$ if and only if there is a finite subset Δ of Γ such that $\Delta \vdash \mathcal{C}$.
3. If $\Delta \vdash \mathcal{C}$, and for each $\mathcal{B}$ in Δ, $\Gamma \vdash \mathcal{B}$, then $\Gamma \vdash \mathcal{C}$.

Assertion 1 represents the fact that if $\mathcal{C}$ is provable from a set

Γ of premisses, then, if we add still more premisses, $\mathscr{C}$ is still provable. Half of assertion 2 follows from 1. The other half is obvious when we notice that any proof of $\mathscr{C}$ from Γ uses only a finite number of premisses from Γ. Assertion 3 is also quite simple: if $\mathscr{C}$ is provable from premisses in Δ, and each premiss in Δ is provable from premisses in Γ, then $\mathscr{C}$ is provable from premisses in Γ.

We now introduce a formal axiomatic theory L for the propositional calculus.

1. The symbols of L are $\neg$, $\Rightarrow$, (,), and the letters A_i with the positive integers i as subscripts: $A_1, A_2, A_3, \ldots$. The symbols $\neg$ and $\Rightarrow$ are called primitive connectives and the letters A_i are called statement letters.

2. a. All statement letters are wfs

b. If $\mathscr{B}$ and $\mathscr{C}$ are wfs, then so are $(\neg\mathscr{B})$ and $(\mathscr{B}\Rightarrow\mathscr{C})$. Thus, a wf of L is just a statement form built up from the statement letter A_i by means of the connectives $\neg$ and $\Rightarrow$

c. An expression is a wf if and only if it can be shown to be a wf on the basis of (a) and (b).

3. If $\mathscr{B}, \mathscr{C}, \mathscr{D}$ are wfs of L, then the following are axioms of L:

(A1) $(\mathscr{B}\Rightarrow(\mathscr{C}\Rightarrow\mathscr{B}))$

(A2) $((\mathscr{B}\Rightarrow(\mathscr{C}\Rightarrow\mathscr{D}))\Rightarrow((\mathscr{B}\Rightarrow\mathscr{C})\Rightarrow(\mathscr{B}\Rightarrow\mathscr{D}))$

(A3) $(((\neg\mathscr{C})\Rightarrow(\neg\mathscr{B}))\Rightarrow(((\neg\mathscr{C})\Rightarrow\mathscr{B})\Rightarrow\mathscr{C}))$

4. The only rule of inference of L is modus ponens: $\mathscr{C}$ is a direct consequence of $\mathscr{B}$ and $(\mathscr{B}\Rightarrow\mathscr{C})$. We shall abbreviate the application of this rule by MP

Notice that the infinite set of axioms of L is given by means of three axiom schemas (A1) – A(3), with each schema standing for an infinite number of axioms. One can easily check for any given wf whether or not it is an axiom; therefore L is axiomatic.

In setting up the system L, it is our intention to obtain as theorems precisely the class of all tautologies.

We introduce other connectives by definition:

(D1) $(\mathcal{B} \wedge \mathcal{C})$ for $\neg(\mathcal{B} \Rightarrow \neg\mathcal{C})$

(D2) $(\mathcal{B} \vee \mathcal{C})$ for $(\neg\mathcal{B}) \Rightarrow \mathcal{C}$

(D3) $(\mathcal{B} \Leftrightarrow \mathcal{C})$ for $(\mathcal{B} \Rightarrow \mathcal{C}) \wedge (\mathcal{C} \Rightarrow \mathcal{B})$

The meaning of (D1), for example, is that, for any wfs $\mathcal{B}$ and $\mathcal{C}$, "$(\mathcal{B} \wedge \mathcal{C})$" is an abbreviation for "$\neg(\mathcal{B} \Rightarrow \neg\mathcal{C})$".

Lemma 1: $\vdash \mathcal{B} \Rightarrow \mathcal{B}$ for all wfs $\mathcal{B}$ in L.

Proof: we shall construct a proof in L of $\mathcal{B} \Rightarrow \mathcal{B}$

1. $(\mathcal{B} \Rightarrow ((\mathcal{B} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{B})) \Rightarrow ((\mathcal{B} \Rightarrow (\mathcal{B} \Rightarrow \mathcal{B})) \Rightarrow (\mathcal{B} \Rightarrow \mathcal{B}))$ Instance of axiom schema (A2)
2. $\mathcal{B} \Rightarrow ((\mathcal{B} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{B})$ Axiom schema (A1)
3. $(\mathcal{B} \Rightarrow (\mathcal{B} \Rightarrow \mathcal{B})) \Rightarrow (\mathcal{B} \Rightarrow \mathcal{B})$ From 1 and 2 by MP
4. $\mathcal{B} \Rightarrow (\mathcal{B} \Rightarrow \mathcal{B})$ Axiom schema (A1)
5. $\mathcal{B} \Rightarrow \mathcal{B}$ From 3 and 4 by MP

Proposition 1.5 Prove: $\mathcal{B} \Rightarrow \mathcal{C},\ \mathcal{C} \Rightarrow \mathcal{D} \vdash \mathcal{B} \Rightarrow \mathcal{D}$ in L

Proof:
1. $\mathcal{C} \Rightarrow \mathcal{D}$ Hypothesis
2. $\mathcal{B} \Rightarrow \mathcal{C}$ Hypothesis
3. $(\mathcal{B} \Rightarrow (\mathcal{C} \Rightarrow \mathcal{D})) \Rightarrow ((\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow (\mathcal{B} \Rightarrow \mathcal{D}))$ Axiom (A2)
4. $(\mathcal{C} \Rightarrow \mathcal{D}) \Rightarrow (\mathcal{B} \Rightarrow (\mathcal{C} \Rightarrow \mathcal{D}))$ Axiom (A1)
5. $\mathcal{B} \Rightarrow (\mathcal{C} \Rightarrow \mathcal{D})$ 1, 4, MP
6. $(\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow (\mathcal{B} \Rightarrow \mathcal{D})$ 3, 5, MP
7. $\mathcal{B} \Rightarrow \mathcal{D}$ 2, 6, MP

Ex. 12. Prove:
a. $\vdash (\neg\mathcal{B} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{B}$ in L
b. $\mathcal{B} \Rightarrow (\mathcal{C} \Rightarrow \mathcal{D}) \vdash \mathcal{C} \Rightarrow (\mathcal{B} \Rightarrow \mathcal{D})$ in L
c. $\vdash (\neg\mathcal{C} \Rightarrow \neg\mathcal{B}) \Rightarrow (\mathcal{B} \Rightarrow \mathcal{C})$ in L

Do it yourself