Proposition 1.7 (Deduction Theorem): If Γ is a set of wfs and $\mathcal{B}$ and $\mathcal{C}$ are wfs, and $\Gamma, \mathcal{B} \vdash \mathcal{C}$, then $\Gamma \vdash \mathcal{B} \Rightarrow \mathcal{C}$. In particular, if $\mathcal{B} \vdash \mathcal{C}$, then $\vdash \mathcal{B} \Rightarrow \mathcal{C}$

Proof: Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ be a proof from $\Gamma \cup \{\mathcal{B}\}$, where $\mathcal{C}_n$ is $\mathcal{C}$. Let us prove, by induction on j, that $\Gamma \vdash \mathcal{B} \Rightarrow \mathcal{C}_j$ for $1 \le j \le n$. First of all, $\mathcal{C}_1$ must be either in Γ or an axiom of L or $\mathcal{B}$ itself. By axiom schema (A1), $\mathcal{C}_1 \Rightarrow (\mathcal{B} \Rightarrow \mathcal{C}_1)$ is an axiom. Hence, in the first two cases, by MP, $\Gamma \vdash \mathcal{B} \Rightarrow \mathcal{C}_1$. For the third case, when $\mathcal{C}_1$ is $\mathcal{B}$, we have $\vdash \mathcal{B} \Rightarrow \mathcal{C}_1$ by Lemma 1, and, therefore, $\Gamma \vdash \mathcal{B} \Rightarrow \mathcal{C}_1$. This takes care of the case $j=1$. Assume now that $\Gamma \vdash \mathcal{B} \Rightarrow \mathcal{C}_k$ for all $k<j$. Either $\mathcal{C}_j$ is an axiom or $\mathcal{C}_j$ is in Γ, or $\mathcal{C}_j$ is $\mathcal{B}$, or $\mathcal{C}_j$ follows by modus ponens from some $\mathcal{C}_l$ and $\mathcal{C}_m$, where $l<j$, $m<j$ and $\mathcal{C}_m$ has the form $\mathcal{C}_l \Rightarrow \mathcal{C}_j$. In the first three cases, $\Gamma \vdash \mathcal{B} \Rightarrow \mathcal{C}_j$ as in the case $j=1$ above. In the last case, we have, by inductive hypothesis, $\Gamma \vdash \mathcal{B} \Rightarrow \mathcal{C}_l$ and $\Gamma \vdash \mathcal{B} \Rightarrow (\mathcal{C}_l \Rightarrow \mathcal{C}_j)$. But, by axiom schema (A2), $\vdash (\mathcal{B} \Rightarrow (\mathcal{C}_l \Rightarrow \mathcal{C}_j)) \Rightarrow ((\mathcal{B} \Rightarrow \mathcal{C}_l) \Rightarrow (\mathcal{B} \Rightarrow \mathcal{C}_j))$. Hence, by MP, $\Gamma \vdash (\mathcal{B} \Rightarrow \mathcal{C}_l) \Rightarrow (\mathcal{B} \Rightarrow \mathcal{C}_j)$, and, again by MP, $\Gamma \vdash \mathcal{B} \Rightarrow \mathcal{C}_j$. Thus, the proof of induction is complete. The case $j=n$ is the desired result.

Notice that, given a deduction of $\mathcal{C}$ from Γ and $\mathcal{B}$, the proof just given enables us to construct a deduction of $\mathcal{B} \Rightarrow \mathcal{C}$ from Γ.

Corollary 1.7.1 a. $\mathcal{B} \Rightarrow \mathcal{C}, \mathcal{C} \Rightarrow \mathcal{D} \vdash \mathcal{B} \Rightarrow \mathcal{D}$

b. $\mathcal{B} \Rightarrow (\mathcal{C} \Rightarrow \mathcal{D}), \mathcal{C} \vdash \mathcal{B} \Rightarrow \mathcal{D}$

Proof: (a).

1.	$\mathcal{B} \Rightarrow \mathcal{C}$	Hyp (Abbreviation for "hypothesis")
2.	$\mathcal{C} \Rightarrow \mathcal{D}$	Hyp
3.	$\mathcal{B}$	Hyp
4.	$\mathcal{C}$	1, 3, MP
5.	$\mathcal{D}$	2, 4, MP

Thus, $\mathcal{B} \Rightarrow \mathcal{C}, \mathcal{C} \Rightarrow \mathcal{D}, \mathcal{B} \vdash \mathcal{D}$. So, by the deduction theorem, $\mathcal{B} \Rightarrow \mathcal{C}, \mathcal{C} \Rightarrow \mathcal{D} \vdash \mathcal{B} \Rightarrow \mathcal{D}$

(b) 1. $\mathcal{B} \Rightarrow (\mathcal{C} \Rightarrow \mathcal{D})$ hyp

2. $\mathcal{B}$ hyp

3. $\mathcal{C} \Rightarrow \mathcal{D}$ 1,2, MP

4. $\mathcal{C}$ hyp

5 $\mathcal{D}$ 3,4, MP

Thus, $\mathcal{B} \Rightarrow (\mathcal{C} \Rightarrow \mathcal{D}), \mathcal{C}, \mathcal{B} \vdash \mathcal{D}$. So, by ~~deductn~~ deduction theorem we get

$$\mathcal{B} \Rightarrow (\mathcal{C} \Rightarrow \mathcal{D}), \mathcal{C} \vdash \mathcal{B} \Rightarrow \mathcal{D}$$

Lemma 2 For any wfs $\mathcal{B}$ and $\mathcal{C}$, the following wfs are theorems of L

a. $\neg\neg\mathcal{B} \Rightarrow \mathcal{B}$

b. $\mathcal{B} \Rightarrow \neg\neg\mathcal{B}$

c. $\neg\mathcal{B} \Rightarrow (\mathcal{B} \Rightarrow \mathcal{C})$

d. $(\neg\mathcal{C} \Rightarrow \neg\mathcal{B}) \Rightarrow (\mathcal{B} \Rightarrow \mathcal{C})$

e. $(\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow (\neg\mathcal{C} \Rightarrow \neg\mathcal{B})$

f. $\mathcal{B} \Rightarrow (\neg\mathcal{C} \Rightarrow \neg(\mathcal{B} \Rightarrow \mathcal{C}))$

g. $(\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow ((\neg\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow \mathcal{C})$

Proof (a) $\vdash \neg\neg\mathcal{B} \Rightarrow \mathcal{B}$

1. $(\neg\mathcal{B} \Rightarrow \neg\neg\mathcal{B}) \Rightarrow ((\neg\mathcal{B} \Rightarrow \neg\mathcal{B}) \Rightarrow \mathcal{B})$ Axiom (A3)

2. $\neg\mathcal{B} \Rightarrow \neg\mathcal{B}$ Lemma 1

3. $(\neg\mathcal{B} \Rightarrow \neg\neg\mathcal{B}) \Rightarrow \mathcal{B}$ 1, 2, Corollary 1.6.1 (b)

4. $\neg\neg\mathcal{B} \Rightarrow (\neg\mathcal{B} \Rightarrow \neg\neg\mathcal{B})$ Axiom (A1)

5. $\neg\neg\mathcal{B} \Rightarrow \mathcal{B}$ 3,4 Corollary 1.6.1 (a)

So, $\vdash \neg\neg\mathcal{B} \Rightarrow \mathcal{B}$

(b) $\vdash$ ~~$\mathcal{B} \Rightarrow$~~ $\neg\neg\mathcal{B} \Rightarrow \mathcal{B}$

1. $(\neg\neg\neg\mathcal{B} \Rightarrow \neg\mathcal{B}) \Rightarrow ((\neg\neg\neg\mathcal{B} \Rightarrow \mathcal{B}) \Rightarrow \neg\neg\mathcal{B})$ Axiom (A3)

2. $\neg\neg\neg\mathcal{B} \Rightarrow \neg\mathcal{B}$ Part (a)

3. $(\neg\neg\neg\mathcal{B} \Rightarrow \mathcal{B}) \Rightarrow \neg\neg\mathcal{B}$ 1,2, MP

4. $\mathcal{B} \Rightarrow (\neg\neg\neg\mathcal{B} \Rightarrow \mathcal{B})$ Axiom (A1)

5. $\mathscr{B} \Rightarrow \neg\neg\mathscr{B}$ 3, 4, Corollary 1.6.1 (a)

So, $\vdash \mathscr{B} \Rightarrow \neg\neg\mathscr{B}$

(c) $\vdash \neg\mathscr{B} \Rightarrow (\mathscr{B} \Rightarrow \mathscr{C})$

1. $\neg\mathscr{B}$ hyp
2. $\mathscr{B}$ hyp
3. $\mathscr{B} \Rightarrow (\neg\mathscr{C} \Rightarrow \mathscr{B})$ Axiom (A1)
4. $\neg\mathscr{B} \Rightarrow (\neg\mathscr{C} \Rightarrow \neg\mathscr{B})$ Axiom (A1)
5. $\neg\mathscr{C} \Rightarrow \mathscr{B}$ 2, 3, MP
6. $\neg\mathscr{C} \Rightarrow \neg\mathscr{B}$ 1, 4, MP
7. $(\neg\mathscr{C} \Rightarrow \neg\mathscr{B}) \Rightarrow ((\neg\mathscr{C} \Rightarrow \mathscr{B}) \Rightarrow \mathscr{C})$ Axiom (A3)
8. $(\neg\mathscr{C} \Rightarrow \mathscr{B}) \Rightarrow \mathscr{C}$ 6, 7, MP
9. $\mathscr{C}$ 5, 8, MP
10. $\neg\mathscr{B}, \mathscr{B} \vdash \mathscr{C}$ 1–9
11. $\neg\mathscr{B} \vdash \mathscr{B} \Rightarrow \mathscr{C}$ 10, Deduction Theorem
12. $\vdash \neg\mathscr{B} \Rightarrow (\mathscr{B} \Rightarrow \mathscr{C})$ 11, Deduction Theorem

Exercise

(d) $\vdash (\neg\mathscr{C} \Rightarrow \neg\mathscr{B}) \Rightarrow (\mathscr{B} \Rightarrow \mathscr{C})$

1. $\neg\mathscr{C} \Rightarrow \neg\mathscr{B}$ Hyp
2. $(\neg\mathscr{C} \Rightarrow \neg\mathscr{B}) \Rightarrow ((\neg\mathscr{C} \Rightarrow \mathscr{B}) \Rightarrow \mathscr{C})$ Axiom (A3)
3. $(\neg\mathscr{C} \Rightarrow \mathscr{B}) \Rightarrow \mathscr{C}$ 1, 2, MP
4. $\mathscr{B} \Rightarrow (\neg\mathscr{C} \Rightarrow \mathscr{B})$ Axiom (A1)
5. $\mathscr{B} \Rightarrow \mathscr{C}$ 3, 4, Corollary 1.6.1 (a)
6. $\neg\mathscr{C} \Rightarrow \neg\mathscr{B} \vdash \mathscr{B} \Rightarrow \mathscr{C}$ 1–5
7. $\vdash (\neg\mathscr{C} \Rightarrow \neg\mathscr{B}) \Rightarrow (\mathscr{B} \Rightarrow \mathscr{C})$ 6, deduction theorem

(e) $\vdash (\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow (\neg\mathscr{C} \Rightarrow \neg\mathscr{B})$

1. $\mathscr{B} \Rightarrow \mathscr{C}$ Hyp.

2. $\neg\neg\mathcal{B} \Rightarrow \mathcal{B}$ part (a)

3. $\neg\neg\mathcal{B} \Rightarrow \mathcal{C}$ 1, 2, Corollary 1.6.1(a)

4. $\mathcal{C} \Rightarrow \neg\neg\mathcal{C}$ ~~part (a)~~ part (b)

5. $\neg\neg\mathcal{B} \Rightarrow \neg\neg\mathcal{C}$ 3, 4, Corollary 1.6.1(a)

6. $(\neg\neg\mathcal{B} \Rightarrow \neg\neg\mathcal{C}) \Rightarrow (\neg\mathcal{C} \Rightarrow \neg\mathcal{B})$ part (d)

~~7. $\neg\mathcal{B} \Rightarrow \neg\mathcal{B}$~~

7. $\neg\mathcal{C} \Rightarrow \neg\mathcal{B}$ 5, 6, MP

8. $\mathcal{B} \Rightarrow \mathcal{C} \vdash \neg\mathcal{C} \Rightarrow \neg\mathcal{B}$ 1–7

9. $\vdash (\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow (\neg\mathcal{C} \Rightarrow \neg\mathcal{B})$ 8, deduction theorem

(f) $\vdash \mathcal{B} \Rightarrow (\neg\mathcal{C} \Rightarrow \neg(\mathcal{B} \Rightarrow \mathcal{C}))$

Clearly, $\mathcal{B}$, $\mathcal{B} \Rightarrow \mathcal{C} \vdash \mathcal{C}$ by MP. Hence $\vdash \mathcal{B} \Rightarrow ((\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow \mathcal{C}))$ by two ~~uses~~ uses of deduction theorem. Now by (e), $\vdash ((\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow \mathcal{C}) \Rightarrow (\neg\mathcal{C} \Rightarrow \neg(\mathcal{B} \Rightarrow \mathcal{C}))$ Hence by Corollary 1.6.1(a), $\vdash \mathcal{B} \Rightarrow (\neg\mathcal{C} \Rightarrow \neg(\mathcal{B} \Rightarrow \mathcal{C}))$.

(g) $\vdash (\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow ((\neg\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow \mathcal{C})$

1. $\mathcal{B} \Rightarrow \mathcal{C}$ Hyp

2. $\neg\mathcal{B} \Rightarrow \mathcal{C}$ Hyp

3. $(\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow (\neg\mathcal{C} \Rightarrow \neg\mathcal{B})$ part (e)

4. $\neg\mathcal{C} \Rightarrow \neg\mathcal{B}$ 1, 3, MP

5. $(\neg\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow (\neg\mathcal{C} \Rightarrow \neg\neg\mathcal{B})$ part (e)

6. $\neg\mathcal{C} \Rightarrow \neg\neg\mathcal{B}$ 2, 5, MP

7. $(\neg\mathcal{C} \Rightarrow \neg\neg\mathcal{B}) \Rightarrow ((\neg\mathcal{C} \Rightarrow \neg\mathcal{B}) \Rightarrow \mathcal{C})$ Axiom (A3)

8. $(\neg\mathcal{C} \Rightarrow \neg\mathcal{B}) \Rightarrow \mathcal{C}$ 6, 7, MP

9. $\mathcal{C}$ 4, 8, MP

10. $\mathcal{B} \Rightarrow \mathcal{C}, \neg\mathcal{B} \Rightarrow \mathcal{C} \vdash \mathcal{C}$ 1–9

11. $\mathcal{B} \Rightarrow \mathcal{C} \vdash (\neg\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow \mathcal{C}$ 10, deduction theorem

12. $\vdash (\mathcal{B} \Rightarrow \mathcal{C}) \vdash ((\neg\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow \mathcal{C})$ 11, deduction theorem

Ex.13 Show that the following wfs are theorems of L:

a. $\mathcal{B} \Rightarrow (\mathcal{B} \vee \mathcal{C})$

b. $\mathcal{B} \Rightarrow (\mathcal{C} \vee \mathcal{B})$

c. $\mathcal{C} \vee \mathcal{B} \Rightarrow \mathcal{B} \vee \mathcal{C}$

d. $\mathcal{B} \wedge \mathcal{C} \Rightarrow \mathcal{B}$

e. $\mathcal{B} \wedge \mathcal{C} \Rightarrow \mathcal{C}$

f. $(\mathcal{B} \Rightarrow \mathcal{D}) \Rightarrow ((\mathcal{C} \Rightarrow \mathcal{D}) \Rightarrow (\mathcal{B} \vee \mathcal{C} \Rightarrow \mathcal{D}))$

g. $((\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow \mathcal{B}) \Rightarrow \mathcal{B}$

h. $\mathcal{B} \Rightarrow (\mathcal{C} \Rightarrow (\mathcal{B} \wedge \mathcal{C}))$

The Soundness Theorem tells us that our natural deduction proofs represent a sound (or correct) system of reasoning.

Proposition 1.8 (Soundness Theorem): Every theorem of L is a tautology

Proof: As an exercise, verify that all the axioms of L are tautologies. (As for example,

$\mathcal{B}$	$\mathcal{C}$	$\mathcal{C} \Rightarrow \mathcal{B}$	$\mathcal{B} \Rightarrow (\mathcal{C} \Rightarrow \mathcal{B})$
T	T	T	T
F	T	F	T
T	F	T	T
F	F	T	T

So, $\mathcal{B} \Rightarrow (\mathcal{C} \Rightarrow \mathcal{B})$ is a tautology, That is, axiom (A1) is a tautology etc.). By proposition 1.2, modus ponens leads from tautologies to other tautologies.

Hence, every theorem of L is a tautology.

Lemma 3. Let $\mathcal{B}$ be a wf and $B_1, B_2, \ldots, B_k$ be the statement letters that occurs in $\mathcal{B}$. For a given assignment of the truth values of $B_1, B_2, \ldots, B_k$, let B_j' be B_j if B_j takes the value T and let B_j' be $\neg B_j$ if B_j takes the value F. Let $\mathcal{B}'$ be $\mathcal{B}$ if $\mathcal{B}$ takes the value T under the assignment, and let $\mathcal{B}'$ be $\neg\mathcal{B}$ if $\mathcal{B}$ takes the value F. Then $B_1', B_2', \ldots, B_k' \vdash \mathcal{B}'$.