For example, let $\mathcal{B}$ be $\neg(\neg A_2 \Rightarrow A_5)$. Then for each row of the truth table

A_2	A_5	$\neg(\neg A_2 \Rightarrow A_5)$
T	T	F
F	T	F
T	F	F
F	F	T

Lemma 3 asserts ~~that~~ a corresponding deducibility relation. For instance, corresponding to the third row there is $A_2, \neg A_5 \vdash \neg\neg(\neg A_2 \Rightarrow A_5)$, and corresponding to the fourth row, $\neg A_2, \neg A_5 \vdash \neg(\neg A_2 \Rightarrow A_5)$.

Proof: The proof is by induction on the number ~~of~~ n of occurences of $\neg$ and $\Rightarrow$ in $\mathcal{B}$. (We assume $\mathcal{B}$ written without abbreviations). If $n=0$, $\mathcal{B}$ is just a statement letter B_1 and then the lemma reduces to $B_1 \vdash B_1$ and $\neg B_1 \vdash \neg B_1$. Assume now that the lemma holds for all $j<n$.

Case 1: $\mathcal{B}$ is $\neg\mathcal{C}$. Then $\mathcal{C}$ has fewer than n occurences of $\neg$ and $\Rightarrow$.

Subcase 1a: Let $\mathcal{C}$ takes the value T under the given truth value assignment. Then $\mathcal{B}$ takes the value F. So, $\mathcal{C}'$ is $\mathcal{C}$ and $\mathcal{B}'$ is $\neg\mathcal{B}$. By the inductive hypothesis applied to $\mathcal{C}$, $B_1', B_2', \ldots, B_k' \vdash \mathcal{C}$.
Then by Lemma 2(b) and MP, $B_1', B_2', \ldots, B_k' \vdash \neg\neg\mathcal{C}$. But $\neg\neg\mathcal{C}$ is $\mathcal{B}'$

Subcase 1b: Let $\mathcal{C}$ takes the value F. Then $\mathcal{B}$ takes the value T. So, $\mathcal{C}'$ is $\neg\mathcal{C}$ and $\mathcal{B}'$ is $\mathcal{B}$. By inductive hypothesis, $B_1', B_2', \cdots B_k' \vdash \neg\mathcal{C}$. But $\neg\mathcal{C}$ is $\mathcal{B}$.

Case 2: $\mathcal{B}$ is $\mathcal{C} \Rightarrow \mathcal{D}$. Then $\mathcal{C}$ and $\mathcal{D}$ have fewer occurences of $\neg$ and $\Rightarrow$ than $\mathcal{B}$. So, by inductive hypothesis, $B_1', B_2', \ldots, B_k' \vdash \mathcal{C}'$ and $B_1', B_2', \ldots, B_k' \vdash \mathcal{D}'$

Subcase 2a $\mathcal{C}$ take the value F. Then $\mathcal{B}$ takes the value T. So, $\mathcal{C}'$ is $\neg\mathcal{C}$ and $\mathcal{B}'$ is $\mathcal{B}$. So, $B_1', B_2', \ldots, B_k' \vdash \neg\mathcal{C}$. By lemma 2(c) and MP, $B_1', B_2', \ldots, B_k' \vdash \mathcal{C} \Rightarrow \mathcal{D}$. But $\mathcal{C} \Rightarrow \mathcal{D}$ is $\mathcal{B}'$.

Subcase 2b: $\mathscr{D}$ takes the value T. Then $\mathscr{B}$ takes the value T. So, $\mathscr{D}'$ is $\mathscr{D}$ and $\mathscr{B}'$ is $\mathscr{B}$. Hence $B_1', B_2', \ldots, B_k' \vdash \mathscr{D}$. Then by axiom (A1) and MP, $B_1', B_2', \ldots, B_k' \vdash \mathscr{C} \Rightarrow \mathscr{D}$. But $\mathscr{C} \Rightarrow \mathscr{D}$ is $\mathscr{B}'$.

Subcase 2c: $\mathscr{C}$ takes the value T and $\mathscr{D}$ takes the value F. Then $\mathscr{B}$ takes the value F. So, $\mathscr{C}'$ is $\mathscr{C}$, $\mathscr{D}'$ is $\neg\mathscr{D}$ and $\mathscr{B}'$ is $\neg\mathscr{B}$. So, $B_1', B_2', \ldots B_k' \vdash \mathscr{C}$ and $B_1', B_2', \ldots B_k' \vdash \neg\mathscr{D}$. Hence, by Lemma 2(f) an MP, $B_1', B_2', \ldots, B_k' \vdash \neg(\mathscr{C} \Rightarrow \mathscr{D})$. But $\neg(\mathscr{C} \Rightarrow \mathscr{D})$ is $\mathscr{B}'$.

[Note: we could have take Subcase 2b: $\mathscr{D}$ takes the value T and $\mathscr{C}$ takes the value T and then there would be no repetition of cases]

Proposition 1.9 (Completeness Theorem)

If a wf of L is a tautology, then it is a theorem of L.

Proof: Assume $\mathscr{B}$ is a tautology, and let $B_1, \ldots, B_k$ be the statement letters in $\mathscr{B}$. For any truth value assignment to $B_1, B_2, \ldots, B_k$, we have by Lemma 3, $B_1', \ldots, B_k' \vdash \mathscr{B}$ ($\mathscr{B}'$ is $\mathscr{B}$ because $\mathscr{B}$ always takes the value T). Hence, when B_k is given the value T, we obtain $B_1', B_2', \ldots, B_{k-1}', B_k \vdash \mathscr{B}$ and B_k is given the value F, we obtain $B_1', B_2', \ldots, B_{k-1}', \neg B_k \vdash \mathscr{B}$. So, by deduction theorem, $B_1', B_2', \ldots, B_{k-1}' \vdash B_k \Rightarrow \mathscr{B}$ and $B_1', B_2', \ldots, B_{k-1}' \vdash \neg B_k \Rightarrow \mathscr{B}$. Then by lemma 2(g) and MP, $B_1', B_2', \ldots, B_{k-1}' \vdash \mathscr{B}$. Similarly, B_{k-1} may be chosen to T or F and, again applying the deduction theorem, lemma 2(g) and MP, we can eliminate B_{k-1}' just as we eliminated B_k'. After K such steps, we finally obtain $\vdash \mathscr{B}$.

Corollary 1.10 If $\mathcal{C}$ be an expression involving the signs $\neg$, $\Rightarrow$, $\wedge$, $\vee$ and $\Leftrightarrow$ that is an abbreviation of a wf $\mathcal{B}$ of L, then $\mathcal{C}$ is a tautology if and only if $\mathcal{B}$ is a theorem of L.

Proof: In the definitions (D1)–(D3) (of Page-31), the abbreviating formulas replace wfs to which they are logically equivalent. Hence, by Proposition 1.4, $\mathcal{B}$ and $\mathcal{C}$ are logically equivalent, and $\mathcal{C}$ is a tautology if and only if $\mathcal{B}$ is a tautology. The corollary now follows from Propositions 1.8 and 1.9.

Definition (consistency): A theory M is said to be consistent if there is no wf $\mathcal{B}$ such that both $\mathcal{B}$ and $\neg\mathcal{B}$ are theorems of M. Otherwise M is said to be inconsistent.

Definition (Absolute consistency): A theory M is said to be absolutely consistent if there is a wf $\mathcal{B}$ which is not a theorem, i.e not all wfs are theorems.

Corollary 1.11 The theory L is consistent

Proof: By Proposition 1.8, every theorem of L is a tautology. The negation of a tautology cannot be a tautology and, therefore, it is impossible for both $\mathcal{B}$ and $\neg\mathcal{B}$ to be theorems of L.

Proposition 1.12 A theory M is consistent if and only if it is absolutely consistent.

Proof: Let M be consistent. Then there are wfs which are not theorem because if $\mathcal{B}$ is a theorem then $\neg\mathcal{B}$ is not a theorem. On the other hand, let M be absolutely consistent. If possible

Let M be inconsistent. So $\exists$ a wf $\mathcal{B}$ such that $\mathcal{B}$ and $\neg\mathcal{B}$ are Theorems of M. ~~By Lemma 2(c)~~ we assume that M is a Theory in which Lemma 2(c) is provable and it has modus ponens as a rule of inference. So here $\vdash \neg\mathcal{B} \Rightarrow (\mathcal{B} \Rightarrow \mathcal{C})$ in M. As $\mathcal{B}$ and $\neg\mathcal{B}$ are both true, so using MP any wf $\mathcal{C}$ is provable, a contradiction that M is absolutely consistent.

So, M is consistent.

Corollary 1.12 The theory L is absolutely consistent.

<u>Consistent set</u> : A set Γ of ~~formulas~~ wfs in the theory L is said to be consistent if there is no wf $\mathcal{B}$ such that $\Gamma \vdash \mathcal{B}$ in L and $\Gamma \vdash \neg\mathcal{B}$. Otherwise, Γ is said to be inconsistent.

maximally Consistent set: A set Γ of ~~formulas~~ wfs in the theory L is said to be maximally consistent if it is consistent and if $\exists$ a consistent set Γ' such that $\Gamma \subset \Gamma'$, then $\Gamma = \Gamma'$. A set Γ is absolutely consistent if $\exists$ a wf $\mathcal{B}$ s.t $\Gamma \nvdash \mathcal{B}$. Γ is also consistent if and only Γ is absolutely consistent.

Using Zorn's Lemma or Tukeys Lemma (both of which are equivalent to axiom of choice) we get a Theorem which is called Lindenbaum's lemma or Theorem.

Proposition 1.13. Every consistent set of wfs can be extended to a maximally consistent set.

~~Proof~~ Lemma 4. The union of a chain of consistent sets ordered by $\subset$, is consistent

Proof: Let $\mathcal{C} = \{\Gamma_i : i \in I\}$ be a chain of consistent sets ordered by $\subset$ and indexed by some set I. We want to show that $\Gamma = \bigcup \mathcal{C}$ is also consistent. Suppose not, then any wf $\mathcal{B}$ is provable from Γ. Let ~~A_1, A_2~~ ... $\mathcal{B}_1, \mathcal{B}_2, \ldots \mathcal{B}_n$ be a deduction of $\mathcal{B}$ (which is $\mathcal{B}_n$) from Γ. So, $\exists$ $\Gamma_{i_1}, \Gamma_{i_2}, \ldots, \Gamma_{i_{n-1}}$ in $\mathcal{C}$ such that $\Gamma_{i_1} \vdash \mathcal{B}_1, \Gamma_{i_2} \vdash \mathcal{B}_2, \ldots, \Gamma_{i_{n-1}} \vdash \mathcal{B}_{n-1}$. Since $\mathcal{C}$ is a chain, take the largest of $\Gamma_{i_1}, \Gamma_{i_2}, \ldots, \Gamma_{i_{n-1}}$, say, Γ_{i_K} so that $\Gamma_K \vdash \mathcal{B}_i$, $i = 1, 2, \ldots, n-1$. This implies $\Gamma_K \vdash \mathcal{B}_n$ or, $\Gamma_{i_K} \vdash \mathcal{B}$, contradicting the assumption that Γ_{i_K} is consistent. So Γ is consistent.

Proof of Proposition 1.13: Let Γ be a consistent set of wfs. Let P be the partially ordered set of all consistent supersets of Γ, ordered by inclusion $\subset$. If $\mathcal{C}$ is a chain of elements in P, then $\bigcup\mathcal{C}$ is consistent by lemma 4, so $\bigcup\mathcal{C} \in P$ as element of $\mathcal{C}$ is a superset of Γ. By Zorn's lemma P has a maximal element. Called it Γ'. To see that Γ' is maximally consistent, suppose Γ is not maximally consistent. So, we can show that $\exists$ a wff $\mathcal{B}$ such that $\Gamma' \nvdash \mathcal{B}$ and $\Gamma' \nvdash \neg\mathcal{B}$. $\Gamma' \nvdash \mathcal{B}$ implies then $\mathcal{B} \notin \Gamma'$ and $\Gamma' \nvdash \neg\mathcal{B}$ implies that $\Gamma' \cup \{\mathcal{B}\}$ is consistent, and therefore in P. So these two results imply that $\Gamma' \cup \{\mathcal{B}\}$ is a consistent proper superset of Γ', contradicting the maximality of Γ' in P. So Γ' is maximally consistent.