2. Predicate Logic or First-order Logic

2.1 Quantifiers

There are various kinds of logical inference that can not be justified on the basis of propositional calculus. ~~condi~~ Consider the following examples:

1. Any friend of Bina is a friend of Rehana.
   Serin is not ~~a~~ ~~friend of~~ Rehana's friend.
   Hence, Serin is not Bina's friend.

2. All human beings are rational.
   Some animals are human beings.
   Hence, some animals are rational.

3. The successor of an even integer is odd.
   2 is an even integer.
   Hence the successor of 2 is odd.

The correction of these inferences rests not only upon the meaning of the truth-functional connectives, but also upon the meaning of such expressions as "any", "all", and "some", and other linguistic constructions.

In order to make the structure of more ~~of~~ complex sentences more transparent, we need to introduce special notation to represent frequently occuring expressions. If $P(x)$ asserts that x has the proper P, then $(\forall x)P(x)$ means that ~~the~~ property P holds for all x or, in other words, that everything has that property P. On the other hand, $(\exists x)P(x)$ means that some x has the property P – that is, there is at least one object having the property P. In $(\forall x)P(x)$, "$\forall x$" is called ~~the~~ a universal quantifier and in $(\exists x)P(x)$, "$\exists x$" is called an existential quantifier. The study of quantifiers and related concepts is the principal subject of this Chapter.

Examples: 1′: Inference 1 above can be represented symbolically:

$$\begin{array}{l} (\forall x)(F(x,m) \Rightarrow F(x,j)) \\ \neg F(p,j) \\ \hline \neg F(p,m) \end{array}$$

Here $F(x,y)$ means that x is a friend of y, while m, j and p denote Bina, Rehana and Serin, ~~&~~ respectively. The horizontal line above "$\neg F(p,m)$" stands for "hence" or "therefore".

2'. Inference 2 becomes: $(\forall x)(H(x) \Rightarrow R(x))$

$$\frac{(\exists x)(A(x) \wedge H(x))}{(\exists x)(A(x) \wedge R(x))}$$

Here H, R, A designate the properties of being human, rational and an animal respectively.

3'. Inference 3 becomes: $(\forall x)(I(x) \wedge E(x) \Rightarrow O(s(x)))$

$$\frac{I(b) \wedge E(b)}{O(s(b))}$$

Here I, E, and O designates respectively the properties of being an integer, even, and odd; s(x) denotes the successor of x and b denotes the integer 2.

Notice that the validity of these inferences does not depend on the particular meanings of F, m, j, p, H, R, A, I, E, O, s and b.

Just as statement forms were used to indicate logical structure dependent upon the logical connectives, so also the form of inferences involving quantifiers, such as inference 1-3, can be represented abstractly, as in 1'-3'. For this purpose, we shall use commas, parentheses, the symbols $\neg$ and $\Rightarrow$ of the propositional Calculus, the universal quantifier symbol $\forall$ and the following groups of symbols:

Individual variables: $x_1, x_2, \ldots x_n, \ldots$

Individual Constants: $a_1, a_2, \ldots, a_n, \ldots$

Predicate letters: A^n_k (n and k are any positive integers)

Function letters: f^n_k (n and k are any positive integers)

The positive integer n that is a superscript of a predicate letter A^n_k or of a function letter f^n_k indicates the number of arguments, whereas the subscript k is just an indexing number to distinguish different predicate or function letters with the same number of arguments.

(For example, in arithmetic both addition and multiplication take two arguments. So, we use two function letter f^2_1 and f^2_2 for addition and multiplication respectively)

In the ~~preceding~~ preceeding examples, x plays the role of an individual variable; m, j, p, and b play the role of individual constants; F is a binary predicate letter (i.e., a predicate letter with two arguments); H, R, A, S, E and O are monadic predicate letters (i.e., predicate letters with one argument); s is a function letter with one argument.

The function letters applied to the variables and individual constants generate the terms:

1. Variables and individual constants are terms
2. If f_k^n is a function letter and $t_1, t_2, \ldots, t_n$ are terms then $f_k^n(t_1, t_2, \ldots, t_n)$ is a term
3. An expression is a term only if it can be shown to be a term on the basis of conditions 1 and 2.

Terms correspond to what in ordinary language are nouns and noun phrases - for examples, "two", "two plus three" and "two plus x".

The predicate letters applied to terms yield the atomic formulas; that is, if A_k^n is a predicate letter and $t_1, t_2, \ldots, t_n$ are terms, then $A_k^n(t_1, t_2, \ldots, t_n)$ is an atomic formula.

The well formed formulas (wfs) of quantification theory are defined as follows:

1. Every atomic formula is a wf
2. If $\mathcal{B}$ and $\mathcal{C}$ are wfs and y is a variable, then $(\neg \mathcal{B})$, $(\mathcal{B} \Rightarrow \mathcal{C})$, and $((\forall y)\mathcal{B})$ are wfs.
3. An expression is a wf only if it can be shown to be a wf on the basis of conditions 1 and 2. ~~In $((\forall y)\mathcal{B})$,~~

In $((\forall y)\mathcal{B})$, "$\mathcal{B}$" is called the scope of the quantifier "$(\forall y)$". Notice that $\mathcal{B}$ need not contain the variable y. In that case, we understand $((\forall y)\mathcal{B})$ to mean the same thing as $\mathcal{B}$.

The expressions $(\mathcal{B} \wedge \mathcal{C})$, $(\mathcal{B} \vee \mathcal{C})$ and $(\mathcal{B} \Leftrightarrow \mathcal{C})$ are defined as in system L (Page-30). It was unnecessary for us to use the symbol $\exists$ as a primitive symbol because we can define existential quantification as follows:

$((\exists x)\mathcal{B})$ stands for $(\neg((\forall x)(\neg \mathcal{B})))$

This definition is faithful to the meaning of quantifiers: $\mathcal{B}(x)$ is true for some x if and only if it is not the case that $\mathcal{B}(x)$ is false for all x.

(We could have taken $\exists$ as primitive and then defined $((\forall x)\mathcal{B})$ as an abbreviation for $(\neg((\exists x)(\neg\mathcal{B})))$, since $\mathcal{B}(x)$ is true for all x if and only if it is not the case that $\mathcal{B}(x)$ is false for some x.)

We remember the following convention: Quantifiers $(\forall y)$ and $(\exists y)$ rank in strength between $\neg, \wedge, \vee$ and $\Rightarrow, \Leftrightarrow$

An occurence of variable x is said to be bound in a wf $\mathcal{B}$ if either it is the occurence of x in a quantifier "$(\forall x)$" in $\mathcal{B}$ or it lies within the scope of a quantifier "$(\forall x)$" in $\mathcal{B}$. Otherwise, the occurence is said to be free in $\mathcal{B}$.

Examples

1. $A_1^2(x_1, x_2)$
2. $A_1^2(x_1,x_2) \Rightarrow (\forall x_1)\, A_1^1(x_1)$
3. $(\forall x_1)\left(A_1^2(x_1,x_2) \Rightarrow (\forall x_1) A_1^1(x_1)\right)$
4. $(\exists x_1)\, A_1^2(x_1,x_2)$

In example 1, the single occurence of x_1 is free. In example 2, the occurence of x_1 in $A_1^2(x_1,x_2)$ is free, but the second and third occurences are bound. In example 3, all occurences of x_1 are bound, and in Example 4 both occurences of x_1 are bound. (Remember that $(\exists x_1) A_1^2(x_1,x_2)$ is an abbreviation ~~for~~ of $\neg(\forall x_1)\neg A_1^2(x_1,x_2)$.) In all four wfs, every occurence of x_2 is free. Notice that, as in Example 2, a variable may have both free and bound occurences in the same wf. Also observe that an occurence of a variable may be bound in some wf $\mathcal{B}$ but free in a subformula of $\mathcal{B}$. For example, the first occurence x_1 is free in the wf of Example 2 but bound in the larger wf of Example 3.

A variable is said to be free (bound) in a wf if it has a free (bound) occurence in $\mathcal{B}$. Thus, a variable may be both free and bound in the same wf; for example, x_1 is free and bound in the wf of Example 2.

Exercise 2.3 Pick out the free and bound occurences of variables in the following wfs:

a. $(\forall x_3)(((\forall x_1)A_1^2(x_1,x_2)) \Rightarrow A_1^2(x_3,a_1))$

b. $(\forall x_2)A_1^2(x_3,x_2) \Rightarrow (\forall x_3)A_1^2(x_3,x_2)$

c. $((\forall x_2)(\exists x_1)A_1^3(x_1,x_2,f_1^2(x_1,x_2))) \vee \neg(\forall x_1)A_1^2(x_2,f_1^1(x_1))$

Exercise 2.4 Indicate the free and bound variables in the wfs of Exercise 2.3.

We shall often indicate that some of the variables $x_{i_1},\dots,x_{i_k}$ are free variables in a wf $\mathcal{B}$ by writing $\mathcal{B}$ as $\mathcal{B}(x_{i_1},\dots,x_{i_k})$. This does not mean that $\mathcal{B}$ contains these variables as free variables, no does it mean that $\mathcal{B}$ does not contain other free variables. This notation is convenient because we can then ~~written~~ agree to write as $\mathcal{B}(t_1,\dots,t_n)$ the result of substituting in $\mathcal{B}$ the terms $t_1,\dots,t_k$ for all free occurences (if any) of $x_{i_1},\dots,x_{i_k}$, respectively.

If $\mathcal{B}$ is a wf and t is a term, then t is said to be free for x_i in $\mathcal{B}$ if no free occurence of x_i in $\mathcal{B}$ lies within the scope of any quantifier $(\forall x_j)$, where x_j is a variable in t. This concept of t being free for x_i in a wf $\mathcal{B}(x_i)$ will have certain technical applications later on. It means that if t is substituted for all free occurences (if any) of x_i in $\mathcal{B}(x_i)$, no occurence of a variable in t becomes a bound occurence in $\mathcal{B}(t)$

Examples

1. The term x_2 is free for x_1 in $A_1^1(x_1)$, but x_2 is not free for x_1 in $(\forall x_2)A_1^1(x_1)$.

2. The term $f_1^2(x_1,x_3)$ is free for x_1 in $(\forall x_2)A_1^2(x_1,x_2) \Rightarrow A_1^1(x_1)$ but is not free for x_1 in $(\exists x_3)(\forall x_2)A_1^2(x_1,x_2) \Rightarrow A_1^1(x_1)$.

The following facts are obvious:

1. A term that contains no variables is free for any variable in any wf.

2. A term t is free for any variable in $\mathcal{B}$ if none of the variables of t is bound in $\mathcal{B}$.

3. x_i is free in any wf

4. Any term is free for x_i in $\mathcal{B}$ if $\mathcal{B}$ contains no free occurrences of x_i.

Exercise 2.5 Is the term $f_1^2(x_1, x_2)$ free for x_1 in the following wfs?

a. $A_1^2(x_1, x_2) \Rightarrow (\forall x_2) A_1^1(x_2)$

b. $((\forall x_2) A_1^2(x_2, a_1)) \vee (\exists x_2) A_1^2(x_1, x_2)$

c. $(\forall x_1) A_1^2(x_1, x_2)$

d. $(\forall x_2) A_1^2(x_1, x_2)$

e. $(\forall x_2) A_1^1(x_2) \Rightarrow A_1^2(x_1, x_2)$

When English sentences are translated into formulas, certain general guidelines will be useful:

1. A sentence of the form "All As are Bs" becomes $(\forall x)(A(x) \Rightarrow B(x))$. For example, Every mathematician loves music is translated as $(\forall x)(M(x) \Rightarrow L(x))$, where $M(x)$ means x is a mathematician and $L(x)$ means x loves music.

2. A sentence of the form "Some As are Bs" becomes $(\exists x)(A(x) \wedge B(x))$. For example, Some New Yorkers are friendly becomes $(\exists x)(N(x) \wedge F(x))$, where $N(x)$ means x is a New Yorker and $F(x)$ means x is friendly.

3. A sentence of the form "No As are Bs" becomes $(\forall x)(A(x) \Rightarrow \neg B(x))$. For example, No philosopher understand politics becomes $(\forall x)(P(x) \Rightarrow \neg U(x))$, where $P(x)$ means x is a philosopher and $U(x)$ means x understands politics.

Let us consider a more complicated example: Some people respect everyone. This can be translated as $(\exists x)(P(x) \wedge (\forall y)(P(y) \Rightarrow R(x,y)))$, where $P(x)$ means x is a person and $R(x,y)$ means x respects y.

Notice that, in informal discussions, to make formulas ~~easil~~ easier to read we may use lower case letters u, v, x, y, z instead of our official notation x_i for individual variables, capital letters $A, B, C, \ldots$ instead of our official notation A_k^n for predicate letters, lower case letters f, g, h, instead of f_k^n for function letters, and lower-case letters $a, b, c, \ldots$ instead of a_i for individual constants.

Exercise 2.6. Translate the following sentences into wfs:

a. Anyone who is persistent can learn logic

b. No politician is honest.