Intuitively, a sequence $s = (s_1, s_2, \dots)$ satisfies a wf $\mathcal{B}$ if and only if, when, for each i, we replace all free occurences of x_i (if any) in $\mathcal{B}$ by a symbol representing s_i, the resulting proposition is true under the given interpretation. Now we can define the notions of truth and falsity of wfs for a given interpretation.

Definitions

1. A wf $\mathcal{B}$ is true for the interpretation M (written $\models_M \mathcal{B}$) if and only if every sequence in Σ satisfies $\mathcal{B}$.
2. $\mathcal{B}$ is said to be false for M if and only if no sequence in Σ satisfies $\mathcal{B}$.
3. An interpretation M is said to be a model for a set Γ of wfs if and only if every wf in Γ is true for M.

The plausibility of our definition of truth will be strengthened by the fact that we can derive all of the following expected properties I–XI of the notions of truth, falsity, and satisfaction.

I. a. $\mathcal{B}$ is false for an interpretation M if and only if $\neg\mathcal{B}$ is true for M.
b. $\mathcal{B}$ is true for M if and only if $\neg\mathcal{B}$ is false for M

II. It is not the case that $\models_M \mathcal{B}$ and $\models_M \neg\mathcal{B}$, that is, no wf can be both true and false for M.

III. If $\models_M \mathcal{B}$ and $\models_M \mathcal{B} \Rightarrow \mathcal{C}$, then $\models_M \mathcal{C}$

IV. $\mathcal{B} \Rightarrow \mathcal{C}$ is false for M if and only if $\models_M \mathcal{B}$ and $\models_M \neg\mathcal{C}$.

V. Consider an interpretation M with domain D.
a. A sequence s satisfies $\mathcal{B} \wedge \mathcal{C}$ if and only if s satisfies $\mathcal{B}$ and s satisfies $\mathcal{C}$.
b. s satisfies $\mathcal{B} \vee \mathcal{C}$ if and only if s satisfies $\mathcal{B}$ or s satisfies $\mathcal{C}$.
c. s satisfies $\mathcal{B} \Leftrightarrow \mathcal{C}$ if and only if s satisfies both $\mathcal{B}$ and $\mathcal{C}$ or s satisfies neither $\mathcal{B}$ nor $\mathcal{C}$.

[Remember that $\mathcal{B} \wedge \mathcal{C}$, $\mathcal{B} \vee \mathcal{C}$, $\mathcal{B} \Leftrightarrow \mathcal{C}$ and $(\exists x_i)\mathcal{B}$ are abbreviations for $\neg(\mathcal{B} \Rightarrow \neg\mathcal{C})$, $\neg\mathcal{B} \Rightarrow \mathcal{C}$, $(\mathcal{B} \Rightarrow \mathcal{C}) \wedge (\mathcal{C} \Rightarrow \mathcal{B})$ and $\neg(\forall x_i)\neg\mathcal{B}$, respectively.]

d. s satisfies $(\exists x_i)\mathcal{B}$ if and only if there is a sequence s' that differs from s in at most ith component such that s' satisfies $\mathcal{B}$. (In other words $s=(s_1, s_2, \cdots, s_i, \cdots)$ satisfies $(\exists x_i)\mathcal{B}$ if and only if there is an element c in the domain D such that the sequence $(s_1, s_2, \cdots, c, \cdots)$ satisfies $\mathcal{B}$.)

VI. $\models_M \mathcal{B}$ if and only if $\models_M (\forall x_i)\mathcal{B}$

we can extend this result in the following way. By the closure of $\mathcal{B}$ we mean the closed wf obtained from $\mathcal{B}$ by prefixing in universal quantifiers those variables, in order of descending subscripts, that are free in $\mathcal{B}$. If $\mathcal{B}$ has no free variables, the closure of $\mathcal{B}$ is defined to be $\mathcal{B}$ itself. For example, if $\mathcal{B}$ is $A_1^2(x_2, x_5) \Rightarrow \neg(\exists x_2) A_1^3(x_1, x_2, x_3)$, its closure is $(\forall x_5)(\forall x_3)(\forall x_2)(\forall x_1)\mathcal{B}$. It follows from (VI) that a wf $\mathcal{B}$ is true if and only if its closure is true

VII. Every instance of a tautology is true for any interpretation. (An instance of a statement form is a wf obtained from the statement form by substituting wfs for all statement letters, with all occurrences of the same statement letter being replaced by the same wf. Thus, an instance of $A_1 \Rightarrow \neg A_2 \vee A_1$ is $A_1^1(x_2) \Rightarrow (\neg(\forall x_1) A_1^1(x_1)) \vee A_1^1(x_2)$)

To prove (VII), show that all instances of the axioms of the system L are true and then use (III) and proposition 1.9.

VIII. If the free variables (if any) of a wf $\mathcal{B}$ occur in the list $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ and if the sequences s and s' have the same components in the i_1th, i_2th, ..., i_kth places, then s satisfies $\mathcal{B}$ if and only if s' satisfies $\mathcal{B}$ [Hint: Use induction on the number of connectives and quantifiers in $\mathcal{B}$. First prove this lemma: If the variables in a term t occur in the list $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ and if s and s' have the same components in the i_1th, i_2th, ..., i_kth places Then $s^*(t) = (s')^*(t)$. In particular, if t contains no variables at all, $s^*(t) = (s')^*(t)$ for any sequence s and s'.]

Although, by (VIII), a particular wf $\mathcal{B}$ with k free variables is essentially satisfied or not only by k-tuples, rather than by denumerable sequences, it is ~~convenient~~ more convenient for a general treatment of satisfaction to deal with infinite rather than finite sequences. If we were to define satisfaction using finite sequences, conditions 3 and 4 of the definition of satisfaction would become much more complicated.

Let $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ be k distinct variables in order of increasing subscripts. Let $\mathcal{B}(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ be a wf that $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ as its only free variables. The set of k-tuples $(b_1, b_2, \ldots, b_k)$ of elements of the domain D such that any sequence with $b_1, b_2, \ldots, b_k$ in its i_1th, i_2th, …, i_kth places, respectively, satisfies $\mathcal{B}(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ is called the relation of the interpretation defined by $\mathcal{B}$. Extending our terminology, we shall say that every k-tuple $(b_1, b_2, \ldots, b_k)$ in this relation satisfies $\mathcal{B}(x_{i_1}, x_{i_2}, \ldots, x_{i_k})$ in the interpretation M; this will be written as $\models_M \mathcal{B}[b_1, b_2, \ldots, b_k]$. This extended notion of satisfaction ~~correspods~~ corresponds to the original intuitive notion.

<u>Examples</u> 1. If the domain D of M is the set of human beings, $A_1^2(x,y)$ is interpreted as x is a brother of y, $A_2^2(x,y)$ is interpreted as x is a parent of y, then the binary relation on D corresponding to the wf $\mathcal{B}(x_1,x_2): (\exists x_3)\left(A_1^2(x_1,x_3) \wedge A_2^2(x_3,x_2)\right)$ is the relation of unclehood. $\models_M \mathcal{B}(b,c)$ when and only when b is an uncle of c

2. If the domain D is the set of positive integers, A_1^2 is interpreted as =, f_1^2 is interpreted as multiplication, and a_1 is interpreted as 1 then the wf $\mathcal{B}(x_1): \neg A_1^2(x_1,a_1) \wedge (\forall x_2)\left((\exists x_3) A_1^2(x_1, f_1^2(x_2, x_3)) \Rightarrow A_1^2(x_2,x_1) \vee A_1^2(x_2,a_1)\right)$ determines the property of being a prime number. Thus $\models_M \mathcal{B}[k]$ if and only if k is a prime number

<u>IX</u>. If $\mathcal{B}$ is a closed wf of a language $\mathcal{L}$, then for any interpretation M either $\models_M \mathcal{B}$ or $\models_M \neg\mathcal{B}$ – that is, either $\mathcal{B}$ is true for M or $\mathcal{B}$ is

false for M. [Hint: Use ($\overline{\text{VIII}}$)]. Of course, $\mathcal{B}$ may be true for some interpretation and false for others. (As an example, consider $A_1^1(a_1)$. If M is an interpretation whose domain is the set of positive integers, A_1^1 is interpreted as the property of being a prime, and the interpretation of a_1 is 2, then $A_1^1(a_1)$ is true. If we change the interpretation by interpreting a_1 as 4, then $A_1^1(a_1)$ becomes false.)

If $\mathcal{B}$ is not closed – that is, if $\mathcal{B}$ contains free variables – $\mathcal{B}$ may be neither true nor false for some interpretation. For example, if $\mathcal{B}$ is $A_1^2(x_1, x_2)$ and we consider an interpretation in which the domain is the set of integers and $A_1^2(y, z)$ is interpreted as $y < z$, then $\mathcal{B}$ is satisfied by only those sequences $s = (s_1, s_2, \ldots)$ of integers in which $s_1 < s_2$. Hence $\mathcal{B}$ is neither true nor false for this interpretation. On the other hand, there are wfs that are not closed but that nevertheless are true or false for every interpretation. A simple example is the wf $A_1^1(x_1) \vee \neg A_1^1(x_1)$, which is true for every interpretation.

$\overline{\text{X}}$. Assume t is free for x_i in $\mathcal{B}(x_i)$. Then $(\forall x_i)\mathcal{B}(x_i) \Rightarrow \mathcal{B}(t)$ is true for all interpretations.

The proof of ($\overline{\text{X}}$) is based upon the following lemmas.

<u>Lemma 1</u> If t and u are terms, s is a sequence in Σ, t' results from t by replacing all occurences of x_i by u, and s' results from s by replacing the ith component of s by $s^*(u)$, then $s^*(t') = (s')^*(t)$.
[Hint: Use induction on the length of t (The length of an expression is the number of occurences of symbols in the expression.)]

<u>Lemma 2</u> Let t be free for x_i in $\mathcal{B}(x_i)$. Then:

a. A sequence $s = (s_1, s_2, \ldots)$ satisfies $\mathcal{B}(t)$ if and only if the sequence s', obtained from s by substituting $s^*(t)$ for s_i in the ith place, satisfies $\mathcal{B}(x_i)$. [Hint: Use induction on the number of connectives and quantifiers in $\mathcal{B}(x_i)$, applying Lemma 1]

b. If $(\forall x_i)\mathcal{B}(x_i)$ is satisfied by the sequence s, then $\mathcal{B}(t)$ is also satisfied by s.

$\overline{\text{XI}}$. If $\mathcal{B}$ does not contain x_i free, then $(\forall x_i)(\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow (\mathcal{B} \Rightarrow (\forall x_i)\mathcal{C})$ is true for all interpretations.

Proof: Assume $\boxed{\text{XI}}$ is not correct. Then $(\forall x_i)(\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow (\mathcal{B} \Rightarrow (\forall x_i)\mathcal{C})$ is not true for some interpretation. By condition 3 of definition of satisfaction, there is a sequence s such that s satisfies $(\forall x_i)(\mathcal{B} \Rightarrow \mathcal{C})$ and s does not satisfy $\mathcal{B} \Rightarrow (\forall x_i)\mathcal{C}$. From the later and Condition 3, s satisfies $\mathcal{B}$ and s does not satisfy $(\forall x_i)\mathcal{C}$. Hence, by condition 4, there is a sequence s', differing from s in at most the ith place, such that s' does not satisfy $\mathcal{C}$. Since x_i is free in neither $(\forall x_i)(\mathcal{B} \Rightarrow \mathcal{C})$ nor $\mathcal{B}$, and since s satisfies both of these wfs, it follows by $(\overline{\text{VIII}})$ that s' also satisfies both $(\forall x_i)(\mathcal{B} \Rightarrow \mathcal{C})$ and $\mathcal{B}$. Since s' satisfies $(\forall x_i)(\mathcal{B} \Rightarrow \mathcal{C})$, it follows by condition 4 that s' satisfies $\mathcal{B} \Rightarrow \mathcal{C}$. Since s' satisfies $\mathcal{B} \Rightarrow \mathcal{C}$ and $\mathcal{B}$, condition 3 implies that s' satisfies $\mathcal{C}$, which contradicts the fact that s' does not satisfy $\mathcal{C}$. Hence $(\overline{\text{XI}})$ is established.

Exercises ~~2000~~

2.10 Prove (I) – (X)

2.11 Prove that a closed wf $\mathcal{B}$ is true for M if and only if $\mathcal{B}$ is satisfied by some sequence s in Σ. (Remember that Σ is the set of denumerable sequences of elements in the domain in M.)

2.12 Find the properties or relations determined by the following wfs and interpretations:

a. $[(\exists u) A_1^2(f_1^2(x,u), y)] \wedge [(\exists v) A_1^2(f_1^2(x,v), z)]$, where the domain D is the set of integers, A_1^2 is $=$, and f_1^2 is multiplication.

b. Here, D is the set of non-negative integers, A_1^2 is $=$, a_1 denotes 0, f_1^2 is addition and f_2^2 is multiplication

i. $[(\exists z)(\neg A_1^2(z, a_1) \wedge A_1^2(f_1^2(x,z), y))]$

ii. $(\exists y) A_1^2(x, f_2^2(y,y))$

c. $(\exists x_3) A_1^2(f_1^2(x_1, x_3), x_2)$, where D is the set of positive integers, A_1^2 is $=$, and f_1^2 is multiplication.