d. $A_1^1(x_1) \wedge (\forall x_2) \neg A_1^2(x_1, x_2)$, where D is the set of all living people, $A_1^1(x)$ mean x is a rich man and $A_1^2(x,y)$ means x is a friend of y.

e. $(\forall x_3)(\exists x_4)\big(A_1^2(f_1^2(x_4,x_3),x_1) \wedge (\exists x_4)(A_1^2(f_1^2(x_4,x_3),x_2)) \Rightarrow A_1^2(x_3,a_1)\big)$, where D is the set of positive integers, A_1^2 is $=$, f_1^2 is multiplication, and a_1 denotes 1.

2.13. For each of the following sentences and interpretations, write a translation into ordinary English and determine its truth and falsity.

a. The domain D is the set of nonnegative integers, A_1^2 is $=$, f_1^2 is addition, f_2^2 is multiplication, a_1 denotes 0, a_2 denotes 1.

i. $(\forall x)(\exists y)\big(A_1^2(x, f_1^2(y,y)) \vee A_1^2(x, f_1^2(f_1^2(y,y), a_2))\big)$

ii. $(\forall x)(\forall y)\big(A_1^2(f_2^2(x,y),a_1) \Rightarrow A_1^2(x,a_1) \vee A_1^2(y,a_1)\big)$

iii. $(\exists y) A_1^2(f_1^2(y,y), a_2)$

b. Here D is the set of integers, A_1^2 is $=$, and f_1^2 is addition

i. $(\forall x_1)(\forall x_2) A_1^2(f_1^2(x_1,x_2), f_1^2(x_2,x_1))$

ii. $(\forall x_1)(\forall x_2)(\forall x_3) A_1^2\big(f_1^2(x_1, f_1^2(x_2,x_3)), f_1^2(f_1^2(x_1,x_2), x_3)\big)$

iii. $(\forall x_1)(\forall x_2)(\exists x_3) A_1^2(f_1^2(x_1,x_3), x_2)$

c. The wfs are same as in part (b), but the domain is the set of positive integers, A_1^2 is $=$, and $f_1^2(x,y)$ is x^y.

d. The domain is the set of rational numbers, A_1^2 is $=$, A_2^2 is $<$, f_1^2 is multiplication, $f_1^1(x)$ is $x+1$ and a_1 denotes 0.

i. $(\exists x) A_1^2(f_1^2(x,x), f_1^1(f_1^1(a_1)))$

ii. $(\forall x)(\forall y)\big(A_2^2(x,y) \Rightarrow (\exists z)(A_2^2(x,z) \wedge A_2^2(z,y))\big)$

iii. $(\forall x)\big(\neg A_1^2(x,a_1) \Rightarrow (\exists y) A_1^2(f_1^2(x,y), f_1^1(a_1))\big)$

e. The domain is the set of nonnegative integers, $A_1^2(u,v)$ means $u \le v$, and $A_1^3(u,v,w)$ means $u+v=w$

i. $(\forall x)(\forall y)(\forall z)\big(A_1^3(x,y,z) \Rightarrow A_1^3(y,x,z)\big)$

ii. $(\forall x)(\forall y)\big(A_1^2(x,y) \Rightarrow A_1^3(x,x,y)\big)$

iii. $(\forall x)(\forall y)\big(A_1^3(x,x,y) \Rightarrow A_1^2(x,y)\big)$

iv. $(\exists x)(\forall y) A_1^3(x,y,y)$

v. $(\exists y)(\forall x) A_1^2(x,y)$

vi. $(\forall x)(\forall y)\left(A_1^2(x,y) \Leftrightarrow (\exists z)A_1^3(x,z,y)\right)$

f. The domain is the set of nonnegative integers, $A_1^2(u,v)$ means $u=v$, $f_1^2(u,v) = u+v$ and $f_2^2(u,v) = u.v$

i. $\cancel{(\forall x)(\forall y)}\,(\forall x)(\exists y)(\exists z)\, A_1^2\left(x, f_1^2\left(f_2^2(y,y), f_2^2(z,z)\right)\right)$

Definition A wf $\mathcal{B}$ is said to be logically valid if and only if $\mathcal{B}$ is true for every interpretation.

$\mathcal{B}$ is said to be satisfiable if and only if there is an interpretation for which $\mathcal{B}$ is satisfied by at least one sequence.

It is obvious that $\mathcal{B}$ is logically valid if ~~only~~ and only if $\neg\mathcal{B}$ is not satisfiable, and $\mathcal{B}$ is satisfiable if and only if $\neg\mathcal{B}$ is not logically valid.

If $\mathcal{B}$ is a closed wf, then we know that $\mathcal{B}$ is either true or false for any given interpretation; that is, $\mathcal{B}$ is satisfied by all sequence or none. Therefore, if $\mathcal{B}$ is closed, then $\mathcal{B}$ is satisfiable if and only if $\mathcal{B}$ is true for some interpretation.

A set Γ of wfs is said to be satisfiable if and only if there is an interpretation in which there is a sequence that satisfies every wf of Γ.

It is impossible for both a wf $\mathcal{B}$ and $\neg\mathcal{B}$ to be logically valid. For, if $\mathcal{B}$ is true for an interpretation, then $\neg\mathcal{B}$ is false for that interpretation.

We say that $\mathcal{B}$ is contradictory if and only if $\mathcal{B}$ is false for every interpretation, or, equivalently, if and only if $\neg\mathcal{B}$ is logically valid.

$\mathcal{B}$ is said to logically imply $\mathcal{C}$ if and only if, in every interpretation, every sequence that satisfies $\mathcal{B}$ also satisfies $\mathcal{C}$. More generally, $\mathcal{C}$ is said to be a logical consequence of a set Γ of wfs if and only if, in every interpretation, every sequence that satisfies every wf in Γ also satisfies $\mathscr{C}$.

$\mathscr{B}$ and $\mathscr{C}$ are said to be logically equivalent if and only if they logically imply each other.

The following assertions are easy consequences of these definitions.

1. $\mathscr{B}$ logically implies $\mathscr{C}$ if and only if $\mathscr{B} \Rightarrow \mathscr{C}$ is logically valid
2. $\mathscr{B}$ and $\mathscr{C}$ are logically equivalent if and only if $\mathscr{B} \Leftrightarrow \mathscr{C}$ is logically valid.
3. If $\mathscr{B}$ logically implies $\mathscr{C}$ and $\mathscr{B}$ is true in a given interpretation, then so is $\mathscr{C}$.
4. If $\mathscr{C}$ is a logical consequence of a set Γ of wfs and all wfs in Γ are true in a given interpretation, then so is $\mathscr{C}$.

Exercise

2.14 Prove assertions 1–4.

Examples

1. Every instance of a tautology is logically valid $(\boxed{\text{VII}})$ (Page-55)
2. If t is free for x in $\mathscr{B}(x)$, then $(\forall x)\mathscr{B}(x) \Rightarrow \mathscr{B}(t)$ is logically valid $(\boxed{\text{X}})$ (Page-57)
3. If $\mathscr{B}$ does not contain x free, then $(\forall x)(\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow (\mathscr{B} \Rightarrow (\forall x)\mathscr{C})$ is logically valid $(\boxed{\text{XI}})$ (Page-58)
4. $\mathscr{B}$ is logically valid if and only if $(\forall y_1)\cdots(\forall y_n)\mathscr{B}$ is logically valid $(\boxed{\text{VI}})$ (Page-55)
5. The wf $(\forall x_2)(\exists x_1)A_1^2(x_1,x_2) \Rightarrow (\exists x_1)(\forall x_2)A_1^2(x_1,x_2)$ is not logically valid.

As a counterexample, let the domain D be the set of integers and let $A_1^2(y,z)$ means $y < z$. Then $(\forall x_2)(\exists x_1)A_1^2(x_1,x_2)$ is true but $(\exists x_1)(\forall x_2)A_1^2(x_1,x_2)$ is false.

Exercise

2.15 Show that the following wfs are not logically valid.

a. $[(\forall x_1)A_1^1(x_1) \Rightarrow (\forall x_1)A_2^1(x_1)] \Rightarrow [(\forall x_1)(A_1^1(x_1) \Rightarrow A_2^1(x_1))]$

b. $[(\forall x_1)(A_1^1(x_1) \vee A_2^1(x_1))] \Rightarrow [(\forall x_1)A_1^1(x_1) \vee (\forall x_1)A_2^1(x_1)]$

2.16 Show that the following wfs are logically valid

a. $\mathscr{B}(t) \Rightarrow (\exists x_i)\mathscr{B}(x_i)$ if t is free for x_i in $\mathscr{B}(x_i)$

b. $(\forall x_i)\mathscr{B} \Rightarrow (\exists x_i)\mathscr{B}$

c. $(\forall x_i)(\forall x_j)\mathscr{B} \Rightarrow (\forall x_j)(\forall x_i)\mathscr{B}$

d. $(\forall x_i)\mathscr{B} \Leftrightarrow \neg(\exists x_i)\neg\mathscr{B}$

e. $(\forall x_i)(\mathscr{B} \Rightarrow \mathscr{C}) \Rightarrow ((\forall x_i)\mathscr{B} \Rightarrow (\forall x_i)\mathscr{C})$

f. $(\forall x_i)\mathscr{B} \wedge (\forall x_i)\mathscr{C} \Leftrightarrow (\forall x_i)(\mathscr{B} \wedge \mathscr{C})$

g. $(\forall x_i)\mathscr{B} \vee (\forall x_i)\mathscr{C} \Rightarrow (\forall x_i)(\mathscr{B} \vee \mathscr{C})$

h. $(\exists x_i)(\exists x_j)\mathscr{B} \Leftrightarrow (\exists x_j)(\exists x_i)\mathscr{B}$

i. $(\exists x_i)(\forall x_j)\mathscr{B} \Rightarrow (\forall x_j)(\exists x_i)\mathscr{B}$

2.17 a. If $\mathscr{B}$ is a closed wf, show that $\mathscr{B}$ logically implies $\mathscr{C}$ if and only if $\mathscr{C}$ is true for every interpretation for which $\mathscr{B}$ is true.

b. Although, by (VI) of Page-55, $(\forall x_1)A_1^1(x_1)$ is true whenever $A_1^1(x_1)$ is true, find an interpretation for which $A_1^1(x_1) \Rightarrow (\forall x_1)A_1^1(x_1)$ is not true (Hence, the hypothesis that $\mathscr{B}$ is a closed wf is essential in (a))

2.18 Prove that, if the free variables of $\mathscr{B}$ are $y_1, \ldots, y_n$, then $\mathscr{B}$ is satisfiable if and only if $(\exists y_1)\ldots(\exists y_n)\mathscr{B}$ is satisfiable.

2.19. Produce counterexamples to show that the following wfs are not logically valid (that is, in each case, find an interpretation for which the wf is not true.)

a. $$[(\forall x)(\forall y)(\forall z)(A_1^2(x,y) \wedge A_1^2(y,z) \Rightarrow A_1^2(x,z)) \wedge (\forall x)\neg A_1^2(x,x)] \Rightarrow (\exists x)(\forall y)\neg A_1^2(x,y)$$

b. $(\forall x)(\exists y)A_1^2(x,y) \Rightarrow (\exists y)A_1^2(y,y)$

c. $(\exists x)(\exists y)A_1^2(x,y) \Rightarrow (\exists y)A_1^2(y,y)$

d. $[(\exists x)A_1^1(x) \Leftrightarrow A_2^1(x)] \Rightarrow (\forall x)(A_1^1(x) \Leftrightarrow A_2^1(x))$

e. $(\exists x)(A_1^1(x) \Rightarrow A_2^1(x)) \Rightarrow ((\exists x)A_1^1(x) \Rightarrow (\exists x)A_2^1(x))$

f. $$[(\forall x)(\forall y)(A_1^2(x,y) \Rightarrow A_1^2(y,x)) \wedge (\forall x)(\forall y)(\forall z)(A_1^2(x,y) \wedge A_1^2(y,z) \Rightarrow A_1^2(x,z))] \Rightarrow (\forall x)A_1^2(x,x)$$

g. $(\exists x)(\forall y)(A_1^2(x,y) \wedge \neg A_1^2(y,x) \Rightarrow [A_1^2(x,x) \Leftrightarrow A_1^2(y,y)])$

h. $$(\forall x)(\forall y)(\forall z)(A_1^2(x,x) \wedge (A_1^2(x,z) \Rightarrow A_1^2(x,y) \vee A_1^2(y,z))) \Rightarrow (\exists y)(\forall z)A_1^2(y,z)$$

i. $(\exists x)(\forall y)(\exists z)((A_1^2(y,z) \Rightarrow A_1^2(x,z)) \Rightarrow (A_1^2(x,x) \Rightarrow A_1^2(y,x)))$

2.20 By introducing appropriate notation, write the sentences of each of the following arguments as wfs and determine whether the argument is correct, that is, determine whether the conclusion is logically implied by the conjunction of the premisses

a. All scientists are neurotic. No vegetarians are neurotic. Therefore, nonvegetarians are scientists.

b. All men are animals. Some animals are carnivorous. Therefore, some men are carnivorous.

c. Any barber in Kolkata shaves exactly those men in Kolkata who do not shave themselves. Hence, there is no barber in Kolkata.

d. For any numbers x, y, z, if $x > y$ and $y > z$, then $x > z$. $x > x$ is false for all number x. Therefore, for any numbers x and y, if $x > y$, then it is not the case that $y > x$.

e. No student in the statistics class is smarter than every student in the logic class. Hence some student in the logic class is smarter than every student in the statistics class.

f. Everyone who is sane can understand mathematics. None of the students of Political science can understand mathematics. No madmen are fit to vote. Hence, none of the students of Political science is fit to vote.

g. For all positive integers x, $x \leq x$. For all positive integers x, y, z, if $x \leq y$ and $y \leq z$, then $x \leq z$. For all positive integers x and y, $x \leq y$ or $y \leq x$. Therefore, there is a positive integer y such that, for all positive integers x, $y \leq x$

h. For any integers x, y, z, if $x > y$, and $y > z$, then $x > z$. $x > x$ is false for all integers x. Therefore, for any integers x and y, if $x > y$ then it is not the case that $y > x$.

2.21 Determine whether the following sets of wfs are compatible – that is, whether their conjunction is satisfiable.

a. $(\exists x)(\exists y) A_1^2(x,y)$

$(\forall x)(\forall y)(\exists z)(A_1^2(x,z) \wedge A_1^2(z,y))$