b. $(\forall x)(\exists y) A_1^2(y, x)$

$(\forall x)(\forall y)\left(A_1^2(x,y) \Rightarrow \neg A_1^2(z,y)\right)$

$(\forall x)(\forall y)(\forall z)\left(A_1^2(x,y) \wedge A_1^2(y,z) \Rightarrow A_1^2(x,z)\right)$

2.22 Determine whether the following wfs are logically valid

a. $\neg(\exists y)(\forall x)\left(A_1^2(x,y) \Leftrightarrow \neg A_1^2(x,x)\right)$

b. $\left[(\exists x) A_1^1(x) \Rightarrow (\exists x) A_2^1(x)\right] \Rightarrow (\exists x)\left(A_1^1(x) \Rightarrow A_2^1(x)\right)$

c. $(\exists x)\left(A_1^1(x) \Rightarrow (\forall y) A_1^1(y)\right)$

d. $(\forall x)\left(A_1^1(x) \vee A_2^1(x)\right) \Rightarrow \left(((\forall x) A_1^1(x)) \vee (\exists x) A_2^1(x)\right)$

e. $(\exists x)(\exists y)\left(A_1^2(x,y) \Rightarrow (\forall z) A_1^2(z,y)\right)$

f. $(\exists x)(\exists y)\left(A_1^1(x) \Rightarrow A_2^1(y)\right) \Rightarrow (\exists x)\left(A_1^1(x) = A_2^1(x)\right)$

g. $(\forall x)\left(A_1^1(x) \Rightarrow A_2^1(x)\right) \Rightarrow \neg(\forall x)\left(A_1^1(x) \Rightarrow \neg A_2^1(x)\right)$

2.23 Exhibit a logically valid wf that is not an instance of a tautology. However, show that any logically valid open wf (that is, a wf without quantifiers) must be an instance of a tautology.

2.24 a. Find a satisfiable closed wf that is not true in any interpretation whose domain has only one member.

b. Find a satisfiable closed wf that is not true in any interpretation whose domain has fewer than three members.

3. First-order Theories

In the case of the propositional calculus, the method of truth tables provides an effective test as to whether any given statement form is a tautology. However, there does not seem to be any effective process for determining whether given wf is logically valid, since, in general, one has to check the truth of a wf for interpretations with arbitrarily large finite or infinite domains. In fact, we shall see later that, according to a plausible definition of "effective", it may actually be proved that there is no effective way to test for logical validity. The axiomatic method, which was a luxury in the study of the propositional calculus, thus appears to be a ~~necessity~~ necessity in the study of wfs involving quantifiers and we therefore turn now to the consideration of first order Theories.

3.1 ~~Logical Axioms~~ Let $\mathcal{L}$ be a first order language. A first order theory ~~If $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are wfs of $\mathcal{L}$, then the foll~~ in the language $\mathcal{L}$ will be a formal theory K whose symbols and wfs ~~are the~~ are the symbols and wfs of $\mathcal{L}$ and whose axioms and rules of inference are specified in the following way. [you must review the axiomatic system of propositional calculus. we shall use the terminology (proof, theorem, consequence, axiomatic, $\vdash \mathcal{B}$ etc.) and notation ($\Gamma \vdash \mathcal{B}$, $\vdash \mathcal{B}$ introduced there.]

The axioms of K are divided into two classes,: the logical axioms and the proper (or nonlogical) axioms.

3.1.1 Logical Axioms

If $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are wfs of $\mathcal{L}$, the following are logical axioms of K:

(A1) $\mathcal{B} \Rightarrow (\mathcal{C} \Rightarrow \mathcal{B})$

(A2) $(\mathcal{B} \Rightarrow (\mathcal{C} \Rightarrow \mathcal{D})) \Rightarrow ((\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow (\mathcal{B} \Rightarrow \mathcal{D}))$

(A3) $(\neg \mathcal{C} \Rightarrow \neg \mathcal{B}) \Rightarrow ((\neg \mathcal{C} \Rightarrow \mathcal{B}) \Rightarrow \mathcal{C})$

(A4) $(\forall x_i)\mathcal{B}(x_i) \Rightarrow \mathcal{B}(t)$ if $\mathcal{B}(x_i)$ is a wf of $\mathcal{L}$ and t is a term of $\mathcal{L}$ that is free for x_i in $\mathcal{B}(x_i)$. Note here that t may be identical with x_i so that all wfs $(\forall x_i)\mathcal{B} \Rightarrow \mathcal{B}$ are axioms by virtue of axiom (A4).

(A5) $(\forall x_i)(\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow (\mathcal{B} \Rightarrow (\forall x_i)\mathcal{C})$ if $\mathcal{B}$ contains no free occurrences of x_i.

3.1.2 Proper axioms

These can not be specified, since they vary from theory to theory. A first order theory in which there are no proper axioms is called a first order predicate calculus.

3.1.3 Rules of Inference

The rules of inference of any first order theory are:

1. Modus ponens : $\mathcal{C}$ follows from $\mathcal{B}$ and $\mathcal{B} \Rightarrow \mathcal{C}$.
2. Generalization : $(\forall x_i)\mathcal{B}$ follows from $\mathcal{B}$.

We shall use the abbreviations MP and Gen, to indicate applications of these rules.

Definition Let K be a first order theory in the language $\mathcal{L}$. By a model of K we mean an interpretation of $\mathcal{L}$ for which all the axioms of K are true.

By (III) and (VI) of Page-54 and Page-55, if the rules of modus ponens and generalization are applied to wfs that are true for a given interpretation, then the results of these applications are also true. Hence every theorem of K is true in every model of K.

As we shall see, the logical axioms are so designed that the logical consequences (in the sense defined on pages 60-61) of the closures of the axioms of K are precisely the theorems of K. In particular, if K is a first order predicate calculus, it turns out that the theorems of K are just those wfs of K that are logically valid.

Examples of first order theories

1. Partial order. Let the language $\mathcal{L}$ have a single predicate letter A_2^2 and no function letters and individual constants. We shall write $x_i < x_j$ instead of $A_2^2(x_i, x_j)$. The theory K has two proper axioms.

a. $(\forall x_1)(\neg\, x_1 < x_1)$ (irreflexivity)

b. $(\forall x_1)(\forall x_2)(\forall x_3)(x_1 < x_2 \wedge x_2 < x_3 \Rightarrow x_1 < x_3)$ (transitivity)

A model of the theory is called a partially ordered structure

2. Group theory. Let the language $\mathcal{L}$ have one predicate letter A_1^2, one function letter f_1^2, and one individual constant a_1. To conform with ordinary notation, we shall write $t = s$ instead of $A_1^2(t, s)$,

$t+s$ instead of $f_1^2(t,s)$ and 0 instead of a_1. The proper axioms of K are:

a. $(\forall x_1)(\forall x_2)(\forall x_3)(x_1+(x_2+x_3) = (x_1+x_2)+x_3)$ (associativity)

b. $(\forall x_1)(0+x_1 = x_1)$ (identity)

c. $(\forall x_1)(\exists x_2)(x_2+x_1 = 0)$ (inverse)

d. $(\forall x_1)(x_1 = x_1)$ (reflexivity of =)

e. $(\forall x_1)(\forall x_2)(x_1 = x_2 \Rightarrow x_2 = x_1)$ (symmetry of =)

f. $(\forall x_1)(\forall x_2)(\forall x_3)(x_1 = x_2 \wedge x_2 = x_3 \Rightarrow x_1 = x_3)$ (transitivity of =)

g. $(\forall x_1)(\forall x_2)(\forall x_3)(x_2 = x_3 \Rightarrow x_1+x_2 = x_1+x_3 \wedge x_2+x_1 = x_3+x_1)$ (substitutivity of =)

A model for this theory, in which the interpretation of = is the identity relation, is called a group. A group is said to be abelian if, in addition, the wf $(\forall x_1)(\forall x_2)(x_1+x_2 = x_2+x_1)$ is true.

Properties of First order theories

All the results in this section refer to an arbitrary first order theory K. Instead of writing $\vdash_K \mathcal{B}$, we shall simply write $\vdash \mathcal{B}$.

Proposition 3.2 Every wf $\mathcal{B}$ of K that is an instance of a tautology is a theorem of K, and it may be proved using only axioms (A1)–(A3) and MP.

Proof: $\mathcal{B}$ arises from a tautology $\mathcal{S}$ by substitution. By proposition 1.9 (page-38), there is a proof of $\mathcal{S}$ in L. In such a proof, make the same substitution of wfs of K for statement letters as were used in obtaining $\mathcal{B}$ from $\mathcal{S}$, and, for all statement letters in the proof that do not occur in $\mathcal{S}$, substitute an wf of K. Then the resulting sequence of wfs is a proof of $\mathcal{B}$, and this proof uses only axiom schemes (A1)–(A3) and MP

The application of proposition 3.2 in a proof will be indicated by writing "Tautology."

Proposition 3.3 Every theorem of a first order predicate calculus is logically valid.

Proof: Axioms (A1)-(A3) are logically valid by property (VII) of the notion of truth (see page 55), and axioms (A4) and (A5) are logically valid by properties (X) and (XI) (see Page 57-58). By properties (III) and (VI), the rules of inference MP and Gen preserve logical validity (see page 54-55). Hence every theorem of a predicate calculus is logically valid.

Example The wf $(\forall x_2)(\exists x_1) A_1^2(x_1, x_2) \Rightarrow (\exists x_1)(\forall x_2) A_1^2(x_1, x_2)$ is not a theorem of any first order predicate calculus, since it is not logically valid (Example 5, Page-61)

Definition A theory K is consistent if no wf $\mathcal{B}$ and its negation $\neg\mathcal{B}$ are both provable in K. A theory is inconsistent if it is not consistent.

Corollary 3.4 Any first order predicate calculus is consistent

Proof: If a wf $\mathcal{B}$ and its negation $\neg\mathcal{B}$ were both theorems of a first order predicate calculus, then by proposition 3.3, both $\mathcal{B}$ and $\neg\mathcal{B}$ would be logically valid, which is impossible.

Notice that, in an inconsistent theory K, every wf $\mathcal{C}$ of K is provable in K. In fact, assume that $\mathcal{B}$ and $\neg\mathcal{B}$ are both provable in K. Since the wf $\mathcal{B} \Rightarrow (\neg\mathcal{B} \Rightarrow \mathcal{C})$ is an instance of a tautology, that wf is, by proposition 3.2, is provable in K. Then two applications of MP would yield $\vdash \mathcal{C}$.

It follows from this remark that, if some wf of a theory K is not a theorem of K, then K is consistent.

The deduction theorem (proposition 1.7, page-31) for the propositional calculus can not be carried over without modification to first order theories. For example, for any wf $\mathcal{B}$, $\mathcal{B} \vdash_K (\forall x_i)\mathcal{B}$, but it is not always the case that $\vdash_K \mathcal{B} \Rightarrow (\forall x_i)\mathcal{B}$.

Consider a domain containing at least two elements c and d. Let K be a predicate calculus and let $\mathcal{B}$ be $A_1^1(x_1)$. Interpret A_1^1 as a property that holds only for c. Then $A_1^1(x_1)$ is satisfied by any sequence $s = (s_1, s_2, \dots)$ in which $s_1 = c$,