but $(\forall x_1) A_1^1(x_1)$ is satisfied by no sequence at all. Hence $A_1^1(x_1) \Rightarrow (\forall x_1) A_1^1(x_1)$ is not true in this interpretation, and so it is not logically valid. Therefore, by ~~the~~ proposition 3.3, $A_1^1(x_1) \Rightarrow (\forall x_1) A_1^1(x_1)$ is not a theorem of K.

A modified, but still useful, form of the deduction theorem may be derived. Let $\mathcal{B}$ be a wf in a set Γ of wfs and assume that we are given a deduction $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n$ from Γ, together with justification for each step in the deduction. We shall say that $\mathcal{D}_i$ depends upon $\mathcal{B}$ in this proof if and only if:

1. $\mathcal{D}_i$ is $\mathcal{B}$ and the justification for $\mathcal{D}_i$ is that it belongs to Γ, or
2. $\mathcal{D}_i$ is justified as a direct consequence by MP or Gen of some preceding wfs of the sequence, where at least one of these preceding wfs depends upon $\mathcal{B}$.

Example $\mathcal{B}, (\forall x_1)\mathcal{B} \Rightarrow \mathcal{C} \vdash (\forall x_1)\mathcal{C}$

$(\mathcal{D}_1)$	$\mathcal{B}$	Hyp
$(\mathcal{D}_2)$	$(\forall x_1)\mathcal{B}$	$(\mathcal{D}_1)$, Gen
$(\mathcal{D}_3)$	$(\forall x_1)\mathcal{B} \Rightarrow \mathcal{C}$	Hyp.
$(\mathcal{D}_4)$	$\mathcal{C}$	$(\mathcal{D}_2)$, $(\mathcal{D}_3)$, MP
$(\mathcal{D}_5)$	$(\forall x_1)\mathcal{C}$	$(\mathcal{D}_4)$, Gen

Here, $(\mathcal{D}_1)$ depends upon $\mathcal{B}$, $(\mathcal{D}_2)$ depends upon $\mathcal{B}$, $(\mathcal{D}_3)$ depends upon $(\forall x_1)\mathcal{B} \Rightarrow \mathcal{C}$, $(\mathcal{D}_4)$ depends upon $\mathcal{B}$ and $(\forall x_1)\mathcal{B} \Rightarrow \mathcal{C}$, and $(\mathcal{D}_5)$ depends upon $\mathcal{B}$ and $(\forall x_1)\mathcal{B} \Rightarrow \mathcal{C}$.

Proposition 3.5 If $\mathcal{C}$ does not depend upon $\mathcal{B}$ in a deduction showing that $\Gamma, \mathcal{B} \vdash \mathcal{C}$, then $\Gamma \vdash \mathcal{C}$.

Proof: Let $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n$ be a deduction of $\mathcal{C}$ from Γ and $\mathcal{B}$, in which $\mathcal{C}$ does not depend upon $\mathcal{B}$. (In this deduction, $\mathcal{D}_n$ is $\mathcal{C}$.) As an inductive hypothesis, let us assume that the proposition is true for all deductions of length less than n. If $\mathcal{C}$

belongs to Γ or is an axiom, then $\Gamma \vdash \mathscr{C}$. If $\mathscr{C}$ is a direct consequence of one or two preceeding wfs by Gen or MP, then since $\mathscr{C}$ does not depend upon $\mathscr{B}$, neither do these preceeding wfs. By the inductive hypothesis, these preceeding wfs are deducible from Γ alone. Consequently, so is $\mathscr{C}$.

Proposition 3.6 (Deduction Theorem)

Assume that, in some deduction showing that $\Gamma, \mathscr{B} \vdash \mathscr{C}$, no application of Gen to a wf that depends upon $\mathscr{B}$ has as its quantified variable a free variable of $\mathscr{B}$. Then $\Gamma \vdash \mathscr{B} \Rightarrow \mathscr{C}$.

Proof: Let $\mathscr{D}_1, \mathscr{D}_2, \ldots, \mathscr{D}_n$ be a deduction of $\mathscr{C}$ from Γ and $\mathscr{B}$, satisfying the assumption of our proposition. (In this deduction $\mathscr{D}_n$ is $\mathscr{C}$.) Let us show by induction that $\Gamma \vdash \mathscr{B} \Rightarrow \mathscr{D}_i$ for each $i \leq n$. If $\mathscr{D}_i$ is an axiom or belongs to Γ, then $\Gamma \vdash \mathscr{B} \Rightarrow \mathscr{D}_i$, since $\mathscr{D}_i \Rightarrow (\mathscr{B} \Rightarrow \mathscr{D}_i)$ is an axiom. If $\mathscr{D}_i$ is $\mathscr{B}$, then $\Gamma \vdash \mathscr{B} \Rightarrow \mathscr{D}_i$, since, by proposition 3.2, $\vdash \mathscr{B} \Rightarrow \mathscr{B}$. If there exist j and k less than i such that $\mathscr{D}_k$ is $\mathscr{D}_j \Rightarrow \mathscr{D}_i$, then, by inductive hypothesis, $\Gamma \vdash \mathscr{B} \Rightarrow \mathscr{D}_j$ and $\Gamma \vdash \mathscr{B} \Rightarrow (\mathscr{D}_j \Rightarrow \mathscr{D}_i)$. Now by axiom (A2), $\vdash (\mathscr{B} \Rightarrow (\mathscr{D}_j \Rightarrow \mathscr{D}_i)) \Rightarrow ((\mathscr{B} \Rightarrow \mathscr{D}_j) \Rightarrow (\mathscr{B} \Rightarrow \mathscr{D}_i))$. Hence by MP twice, $\Gamma \vdash \mathscr{B} \Rightarrow \mathscr{D}_i$. Finally, suppose there is some $j < i$ such that $\mathscr{D}_i$ is $(\forall x_k)\mathscr{D}_j$. By inductive hypothesis, $\Gamma \vdash \mathscr{B} \Rightarrow \mathscr{D}_j$, and by the hypothesis of the Theorem, either $\mathscr{D}_j$ does not depend upon $\mathscr{B}$ or x_k is not a free variable of $\mathscr{B}$. If $\mathscr{D}_j$ does not depend upon $\mathscr{B}$, then by proposition 3.5, $\Gamma \vdash \mathscr{D}_j$ and consequently, by Gen, $\Gamma \vdash (\forall x_k)\mathscr{D}_j$. Thus $\Gamma \vdash \mathscr{D}_i$. Now by axiom (A1), $\vdash \mathscr{D}_i \Rightarrow (\mathscr{B} \Rightarrow \mathscr{D}_i)$. So, $\Gamma \vdash \mathscr{B} \Rightarrow \mathscr{D}_i$ by MP. If, on the other hand, x_k is not a free variable of $\mathscr{B}$, then by axiom (A5), $\vdash (\forall x_k)(\mathscr{B} \Rightarrow \mathscr{D}_j) \Rightarrow (\mathscr{B} \Rightarrow (\forall x_k)\mathscr{D}_j)$. Since $\Gamma \vdash \mathscr{B} \Rightarrow \mathscr{D}_j$, we have by Gen, $\Gamma \vdash (\forall x_k)(\mathscr{B} \Rightarrow \mathscr{D}_j)$, and so, by MP,

$\Gamma \vdash \mathcal{B} \Rightarrow (\forall x_k)\mathcal{D}_j$; that is, $\Gamma \vdash \mathcal{B} \Rightarrow \mathcal{D}_i$. This completes the induction, and our proposition is just the special case $i = n$.

The hypothesis of Proposition 3.6 is rather cumbersome; the following weaker corollaries often prove to be more useful.

Corollary 3.7 If a deduction showing that $\Gamma, \mathcal{B} \vdash \mathcal{C}$ involves no application of Gen of which the quantified variables is free in $\mathcal{B}$, then $\Gamma \vdash \mathcal{B} \Rightarrow \mathcal{C}$.

Corollary 3.8 If $\mathcal{B}$ be a closed wf and $\Gamma, \mathcal{B} \vdash \mathcal{C}$, then $\Gamma \vdash \mathcal{B} \Rightarrow \mathcal{C}$.

Extension of Propositions 3.5 – 3.8

In proposition 3.5–3.8, the following additional conclusion can be drawn from the proofs. The new proof of $\Gamma \vdash \mathcal{B} \Rightarrow \mathcal{C}$ (in Proposition 3.5, of $\Gamma \vdash \mathcal{C}$) involves an application of Gen to a wf depending upon a wf $\mathcal{E}$ of Γ only if there is an application of ~~Gen~~ Gen in the given proof of $\Gamma, \mathcal{B} \vdash \mathcal{C}$ that involves the same quantified variable and is applied to a wf that depends upon ~~$\mathcal{E}$~~ $\mathcal{E}$. (In the proof of Proposition 3.6, one should observe that $\mathcal{D}_j$ depends upon a premiss $\mathcal{E}$ of Γ in the original proof if and only if $\mathcal{B} \Rightarrow \mathcal{D}_j$ depends upon $\mathcal{E}$ in the new proof.)

This supplementary conclusion will be useful when we wish to apply the deduction theorem several times in a row to a given deduction – for example, to obtain $\Gamma \vdash \mathcal{D} \Rightarrow (\mathcal{B} \Rightarrow \mathcal{C})$ from $\Gamma, \mathcal{D}, \mathcal{B} \vdash \mathcal{C}$, from now on, it is to be considered an integral part of the statements of propositions 3.5 – 3.8.

Example $\vdash (\forall x_1)(\forall x_2)\mathcal{B} \Rightarrow (\forall x_2)(\forall x_1)\mathcal{B}$

Proof:

1. $(\forall x_1)(\forall x_2)\mathcal{B}$ Hyp
2. $(\forall x_1)(\forall x_2)\mathcal{B} \Rightarrow (\forall x_2)\mathcal{B}$ (A4)
3. $(\forall x_2)\mathcal{B}$ 1, 2, MP
4. $(\forall x_2)\mathcal{B} \Rightarrow \mathcal{B}$ (A4)
5. $\mathcal{B}$ 3, 4, MP
6. $(\forall x_1)\mathcal{B}$ 5, Gen
7. $(\forall x_2)(\forall x_1)\mathcal{B}$ 6, Gen.

Thus by 1–7, we have $(\forall x_1)(\forall x_2)\mathcal{B} \vdash (\forall x_2)(\forall x_1)\mathcal{B}$, where, in the deduction, no application of Gen has as a quantified variable a free variable of $(\forall x_1)(\forall x_2)\mathcal{B}$. Hence by corollary 3.7,

$$\vdash (\forall x_1)(\forall x_2)\mathcal{B} \Rightarrow (\forall x_2)(\forall x_1)\mathcal{B}.$$

Exercise

3.9 Derive the following theorems

a. $\vdash (\forall x)(\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow ((\forall x)\mathcal{B} \Rightarrow (\forall x)\mathcal{C})$

b. $\vdash (\forall x)(\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow ((\exists x)\mathcal{B} \Rightarrow (\exists x)\mathcal{C})$

c. $\vdash (\forall x)(\mathcal{B} \wedge \mathcal{C}) \Leftrightarrow (\forall x)\mathcal{B} \wedge (\forall x)\mathcal{C}$

d. $\vdash (\forall y_1)\cdots(\forall y_n)\mathcal{B} \Rightarrow \mathcal{B}$

e. $\vdash \neg(\forall x)\mathcal{B} \Rightarrow (\exists x)\neg\mathcal{B}$

Additional theorems and Derived Rules

For the sake of smoothness in working with particular theories later, we shall introduce various techniques for constructing proofs. In this section, it is assumed that we are dealing with an arbitrary theory K.

Often one wants to $\mathcal{B}(t)$ from $(\forall x)\mathcal{B}(x)$, where t is a term free for x in $\mathcal{B}(x)$. This is allowed by the following derived rules.

3.10 Particularization Rule A4

If t is free for x in $\mathcal{B}(x)$, then $(\forall x)\mathcal{B}(x) \vdash \mathcal{B}(t)$

Proof: From $(\forall x)\mathcal{B}(x)$ and the instance $(\forall x)\mathcal{B}(x) \Rightarrow \mathcal{B}(t)$ of axiom (A4), we obtain $\mathcal{B}(t)$ by modus ponens.

Since x is free in $\mathcal{B}(x)$, a special case of rule A4 is $(\forall x)\mathcal{B} \vdash \mathcal{B}$

There is another very useful derived rule, which is essentially the contrapositive of rule A4.

3.11 Existential Rule E4

Let t be a term that is free for x in a wf $\mathcal{B}(x, t)$, and let $\mathcal{B}(t, t)$ arise from $\mathcal{B}(x, t)$ by replacing all free

occurrences of x by t. ($\mathcal{B}(x,t)$ may or may not contain occurrences of t.)
Then, $\mathcal{B}(t,t) \vdash (\exists x)\mathcal{B}(x,t)$

Proof: It suffices to show that $\vdash \mathcal{B}(t,t) \Rightarrow (\exists x)\mathcal{B}(x,t)$. But, by axiom (A4), $\vdash (\forall x)\neg\mathcal{B}(x,t) \Rightarrow \neg\mathcal{B}(t,t)$. Hence by tautology ~~$A \Rightarrow \neg B$~~ $(A \Rightarrow \neg B) \Rightarrow (B \Rightarrow \neg A)$ and MP, $\vdash \mathcal{B}(t,t) \Rightarrow \neg(\forall x)\neg\mathcal{B}(x,t)$, which in abbreviated form, is $\vdash \mathcal{B}(t,t) \Rightarrow (\exists x)\mathcal{B}(x,t)$

A special case of rule E4 is $\mathcal{B}(t) \vdash (\exists x)\mathcal{B}(x)$, whenever t is free for x in $\mathcal{B}(x)$. In particular, when t is x itself, $\mathcal{B}(x) \vdash (\exists x)\mathcal{B}(x)$

Example $\vdash (\forall x)\mathcal{B} \Rightarrow (\exists x)\mathcal{B}$

1. $(\forall x)\mathcal{B}$	Hyp
2. $\mathcal{B}$	1, rule A4
3. $(\exists x)\mathcal{B}$	2, rule E4
4. $(\forall x)\mathcal{B} \vdash (\exists x)\mathcal{B}$	1-3
5. $\vdash (\forall x)\mathcal{B} \Rightarrow (\exists x)\mathcal{B}$	1-4, Corollary 3.7

The following derived rules are extremely useful.

Negation elimination :	$\neg\neg\mathcal{B} \vdash \mathcal{B}$
Negation introduction :	$\mathcal{B} \vdash \neg\neg\mathcal{B}$
Conjunction elimination :	$\mathcal{B} \wedge \mathcal{C} \vdash \mathcal{B}$
	$\mathcal{B} \wedge \mathcal{C} \vdash \mathcal{C}$
	$\neg(\mathcal{B} \wedge \mathcal{C}) \vdash \neg\mathcal{B} \vee \neg\mathcal{C}$
Conjunction introduction :	$\mathcal{B}, \mathcal{C} \vdash \mathcal{B} \wedge \mathcal{C}$
Disjunction elimination :	$\mathcal{B} \vee \mathcal{C}, \neg\mathcal{B} \vdash \mathcal{C}$
	$\mathcal{B} \vee \mathcal{C}, \neg\mathcal{C} \vdash \mathcal{B}$
	$\neg(\mathcal{B} \vee \mathcal{C}) \vdash \neg\mathcal{B} \wedge \neg\mathcal{C}$
	$\mathcal{B} \Rightarrow \mathcal{D}, \mathcal{C} \Rightarrow \mathcal{D}, \mathcal{B} \vee \mathcal{C} \vdash \mathcal{D}$
Disjunction introduction :	$\mathcal{B} \vdash \mathcal{B} \vee \mathcal{C}$
	$\mathcal{C} \vdash \mathcal{B} \vee \mathcal{C}$
Conditional elimination :	$\mathcal{B} \Rightarrow \mathcal{C}, \neg\mathcal{C} \vdash \neg\mathcal{B}$
	$\mathcal{B} \Rightarrow \neg\mathcal{C}, \mathcal{C} \vdash \neg\mathcal{B}$
	$\neg\mathcal{B} \Rightarrow \mathcal{C}, \neg\mathcal{C} \vdash \mathcal{B}$
	$\neg\mathcal{B} \Rightarrow \neg\mathcal{C}, \mathcal{C} \vdash \mathcal{B}$
	$\neg(\mathcal{B} \Rightarrow \mathcal{C}) \vdash \mathcal{B}$
	$\neg(\mathcal{B} \Rightarrow \mathcal{C}) \vdash \neg\mathcal{C}$