Conditional introduction : $\mathcal{B}, \neg\mathcal{C} \vdash \neg(\mathcal{B} \Rightarrow \mathcal{C})$

Conditional contrapositive : $\mathcal{B} \Rightarrow \mathcal{C} \vdash \neg\mathcal{C} \Rightarrow \neg\mathcal{B}$

$\neg\mathcal{C} \Rightarrow \neg\mathcal{B} \vdash \mathcal{B} \Rightarrow \mathcal{C}$

Biconditional elimination : $\mathcal{B} \Leftrightarrow \mathcal{C}, \mathcal{B} \vdash \mathcal{C}$, $\mathcal{B} \Leftrightarrow \mathcal{C}, \neg\mathcal{B} \vdash \neg\mathcal{C}$

$\mathcal{B} \Leftrightarrow \mathcal{C}, \mathcal{C} \vdash \mathcal{B}$, $\mathcal{B} \Leftrightarrow \mathcal{C}, \neg\mathcal{C} \vdash \neg\mathcal{B}$

$\mathcal{B} \Leftrightarrow \mathcal{C} \vdash \mathcal{B} \Rightarrow \mathcal{C}$, $\mathcal{B} \Leftrightarrow \mathcal{C} \vdash \mathcal{C} \Rightarrow \mathcal{B}$

Biconditional introduction : $\mathcal{B} \Rightarrow \mathcal{C}, \mathcal{C} \Rightarrow \mathcal{B} \vdash \mathcal{B} \Leftrightarrow \mathcal{C}$

Biconditional negation : $\mathcal{B} \Leftrightarrow \mathcal{C} \vdash \neg\mathcal{B} \Leftrightarrow \neg\mathcal{C}$

$\neg\mathcal{B} \Leftrightarrow \neg\mathcal{C} \vdash \mathcal{B} \Leftrightarrow \mathcal{C}$

Proof by contradiction : If a proof of $\Gamma, \neg\mathcal{B} \vdash \mathcal{C} \wedge \neg\mathcal{C}$ involves no application of Gen using a variable free in $\mathcal{B}$, then $\Gamma \vdash \mathcal{B}$. (Similarly one obtains $\Gamma \vdash \neg\mathcal{B}$ from $\Gamma, \mathcal{B} \vdash \mathcal{C} \wedge \neg\mathcal{C}$.)

Exercises

3.10 Justify the derived rules listed above

3.11 Prove the following :

a. $\vdash (\forall x)(\forall y) A_1^2(x,y) \Rightarrow (\forall x) A_1^2(x,x)$

b. $\vdash [(\forall x)\mathcal{B}] \vee [(\forall x)\mathcal{C}] \Rightarrow (\forall x)(\mathcal{B} \vee \mathcal{C})$

c. $\vdash \neg(\exists x)\mathcal{B} \Rightarrow (\forall x)\neg\mathcal{B}$

d. $\vdash (\forall x)\mathcal{B} \Rightarrow (\forall x)(\mathcal{B} \vee \mathcal{C})$

e. $\vdash (\forall x)(\forall y)\left(A_1^2(x,y) \Rightarrow \neg A_1^2(y,x)\right) \Rightarrow (\forall x)\neg A_1^2(x,x)$

f. $\vdash [(\exists x)\mathcal{B} \Rightarrow (\forall x)\mathcal{C}] \Rightarrow (\forall x)(\mathcal{B} \Rightarrow \mathcal{C})$

g. $\vdash (\forall x)(\mathcal{B} \vee \mathcal{C}) \Rightarrow [(\forall x)\mathcal{B}] \vee (\exists x)\mathcal{C}$

h. $\vdash (\forall x)\left(A_1^2(x,x) \Rightarrow (\exists y) A_1^2(x,y)\right)$

i. $\vdash (\forall x)(\mathcal{B} \Rightarrow \mathcal{C}) \Rightarrow [(\forall x)\neg\mathcal{C} \Rightarrow (\forall x)\neg\mathcal{B}]$

j. $\vdash (\exists y)[A_1^1(y) \Rightarrow (\forall y) A_1^1(y)]$

k. $\vdash (\exists x) A_1^2(x,x) \Rightarrow (\exists x)(\exists y) A_1^2(x,y)$

3.12 Assume that $\mathcal{B}$ and $\mathcal{C}$ are wfs and x is not free in $\mathcal{B}$. Prove the following

a. $\vdash \mathcal{B} \Rightarrow (\forall x)\mathcal{B}$

b. $\vdash (\exists x)\mathscr{B} \Rightarrow \mathscr{B}$

c. $\vdash (\mathscr{B} \Rightarrow (\forall x)\mathscr{C}) \Leftrightarrow (\forall x)(\mathscr{B} \Rightarrow \mathscr{C})$

d. $\vdash ((\exists x)\mathscr{C} \Rightarrow \mathscr{B}) \Leftrightarrow (\forall x)(\mathscr{C} \Rightarrow \mathscr{B})$

We need a derived rule that will allow us to replace a part $\mathscr{C}$ of a wf $\mathscr{B}$ by a wf that provably equivalent to $\mathscr{C}$. For this purpose, we must prove the following auxiliary result.

<u>Lemma 3.12</u> For any wfs $\mathscr{B}$ and $\mathscr{C}$, $\vdash (\forall x)(\mathscr{B} \Leftrightarrow \mathscr{C}) \Rightarrow ((\forall x)\mathscr{B} \Leftrightarrow (\forall x)\mathscr{C})$

Proof:

1. $(\forall x)(\mathscr{B} \Rightarrow \mathscr{C})$	Hyp
2. $(\forall x)\mathscr{B}$	Hyp
3. $\mathscr{B} \Leftrightarrow \mathscr{C}$	1, rule A4
4. $\mathscr{B}$	2, rule A4
5. $\mathscr{C}$	3, 4, biconditional elimination
6. $(\forall x)\mathscr{C}$	5, Gen
7. $(\forall x)(\mathscr{B} \Leftrightarrow \mathscr{C}), (\forall x)\mathscr{B} \vdash (\forall x)\mathscr{C}$	1–6
8. $(\forall x)(\mathscr{B} \Leftrightarrow \mathscr{C}) \vdash (\forall x)\mathscr{B} \Rightarrow (\forall x)\mathscr{C}$	1–7, Corollary 3.7
9. $(\forall x)(\mathscr{B} \Leftrightarrow \mathscr{C}) \vdash (\forall x)\mathscr{C} \Rightarrow (\forall x)\mathscr{B}$	Proof like that of 8
10. $(\forall x)(\mathscr{B} \Leftrightarrow \mathscr{C}) \vdash (\forall x)\mathscr{B} \Leftrightarrow (\forall x)\mathscr{C}$	8, 9, Biconditional introduction
11. $\vdash (\forall x)(\mathscr{B} \Leftrightarrow \mathscr{C}) \Rightarrow ((\forall x)\mathscr{B} \Leftrightarrow (\forall x)\mathscr{C})$	1–10, Corollary 3.7

Proposition 3.13

If $\mathscr{C}$ is a subformula of $\mathscr{B}$, $\mathscr{B}'$ is the result of replacing zero or more occurrences of $\mathscr{C}$ in $\mathscr{B}$ by a wf $\mathscr{D}$, and every free variable of $\mathscr{C}$ or $\mathscr{D}$ that is also a bound variable of $\mathscr{B}$ occurs in the list $y_1, y_2, \ldots, y_k$, then:

a. $\vdash [(\forall y_1) \ldots (\forall y_k)(\mathscr{C} \Leftrightarrow \mathscr{D})] \Rightarrow (\mathscr{B} \Leftrightarrow \mathscr{B}')$ [Equivalence Theorem]

b. If $\vdash \mathscr{C} \Leftrightarrow \mathscr{D}$, then $\vdash \mathscr{B} \Rightarrow \mathscr{B}'$ [Replacement theorem]

c. If $\vdash \mathscr{C} \Leftrightarrow \mathscr{D}$ and $\vdash \mathscr{B}$, then $\vdash \mathscr{B}'$

Proof: a. We use induction on the number of connectives and quantifiers in $\mathcal{B}$. Note that, if zero occurences are replaced, $\mathcal{B}'$ is $\mathcal{B}$ and the wf to be proved is an instance of the tautology $A \Rightarrow (B \Leftrightarrow B)$. Note also that, if $\mathcal{C}$ is identical with $\mathcal{B}$ and this occurence of $\mathcal{C}$ is replaced by $\mathcal{D}$, the wf to be proved, $[(\forall y_1)\cdots(\forall y_k)(\mathcal{C} \Leftrightarrow \mathcal{D})] \Rightarrow (\mathcal{B} \Rightarrow \mathcal{B}')$, is derivable by Exercise 3.9(a) (Page-72). Thus, we may assume that $\mathcal{C}$ is a proper part of $\mathcal{B}$ and that at least one occurence of $\mathcal{C}$ is replaced. Our inductive hypothesis is that the result holds for all wfs with fewer connectives and quantifiers than $\mathcal{B}$.

Case 1. $\mathcal{B}$ is an atomic wf. Then $\mathcal{C}$ can not be a proper part of $\mathcal{B}$.

Case 2. $\mathcal{B}$ is $\neg \mathcal{E}$. Let $\mathcal{B}'$ be $\neg \mathcal{E}'$. By inductive hypothesis, $\vdash [(\forall y_1)\cdots(\forall y_k)(\mathcal{C} \Leftrightarrow \mathcal{D})] \Rightarrow (\mathcal{E} \Leftrightarrow \mathcal{E}')$. Hence, by a suitable instance of the tautology $(C \Rightarrow (A \Leftrightarrow B)) \Rightarrow (C \Rightarrow (\neg A \Leftrightarrow \neg B))$ and MP, we obtain $\vdash [(\forall y_1)\cdots(\forall y_k)(\mathcal{C} \Leftrightarrow \mathcal{D})] \Rightarrow (\mathcal{B} \Rightarrow \mathcal{B}')$.

Case 3. $\mathcal{B}$ is $\mathcal{E} \Rightarrow \mathcal{F}$. Let $\mathcal{B}'$ be $\mathcal{E}' \Rightarrow \mathcal{F}'$. By inductive hypothesis,
$\vdash [(\forall y_1)\cdots(\forall y_k)(\mathcal{C} \Leftrightarrow \mathcal{D})] \Rightarrow (\mathcal{E} \Leftrightarrow \mathcal{E}')$ and $\vdash [(\forall y_1)\cdots(\forall y_k)(\mathcal{C} \Leftrightarrow \mathcal{D})] \Rightarrow (\mathcal{F} \Leftrightarrow \mathcal{F}')$.
Using a suitable instance of the tautology
$(A \Rightarrow (B \Leftrightarrow C)) \wedge (A \Rightarrow (D \Leftrightarrow E)) \Rightarrow (A \Rightarrow [(B \Rightarrow D) \Leftrightarrow (C \Rightarrow E)])$
we obtain $\vdash [(\forall y_1)\cdots(\forall y_k)(\mathcal{C} \Leftrightarrow \mathcal{D})] \Rightarrow (\mathcal{B} \Leftrightarrow \mathcal{B}')$

Case 4. $\mathcal{B}$ is $(\forall x)\mathcal{E}$. Let $\mathcal{B}'$ be $(\forall x)\mathcal{E}'$. By inductive hypothesis, $\vdash [(\forall y_1)\cdots(\forall y_k)(\mathcal{C} \Leftrightarrow \mathcal{D})] \Rightarrow (\mathcal{E} \Leftrightarrow \mathcal{E}')$. Now, x does not occur free in $(\forall y_1)\cdots(\forall y_k)(\mathcal{C} \Leftrightarrow \mathcal{D})$ because, if it did, it would be free in $\mathcal{C}$ or $\mathcal{D}$ and, since it is bound in $\mathcal{B}$, it would be one of $y_1, y_2, \ldots, y_k$ and it would not be free in $(\forall y_1)\cdots(\forall y_k)(\mathcal{C} \Leftrightarrow \mathcal{D})$, Hence using axiom (A5), we obtain
$\vdash (\forall y_1)\cdots(\forall y_k)(\mathcal{C} \Leftrightarrow \mathcal{D}) \Rightarrow (\forall x)(\mathcal{E} \Leftrightarrow \mathcal{E}')$. However, by Lemma 3.12,
$\vdash (\forall x)(\mathcal{E} \Leftrightarrow \mathcal{E}') \Rightarrow ((\forall x)\mathcal{E} \Leftrightarrow (\forall x)\mathcal{E}')$. Then by a suitable tautology and MP,
$\vdash [(\forall y_1)\cdots(\forall y_k)(\mathcal{C} \Leftrightarrow \mathcal{D})] \Rightarrow (\mathcal{B} \Leftrightarrow \mathcal{B}')$.

b. From $\vdash \mathcal{C} \Leftrightarrow \mathcal{D}$, by several applications of Gen, we obtain
$\vdash (\forall y_1)\cdots(\forall y_k)(\mathcal{C} \Leftrightarrow \mathcal{D})$. Then by (a) and MP, $\vdash \mathcal{B} \Leftrightarrow \mathcal{B}'$

c. Use part (b) and biconditional elimination.

Exercises

3.13 Prove the following

a. $\vdash (\exists x)\neg\mathcal{B} \Leftrightarrow \neg(\forall x)\mathcal{B}$

b. $\vdash (\forall x)\mathcal{B} \Leftrightarrow \neg(\exists x)\neg\mathcal{B}$

c. $\vdash (\exists x)(\mathcal{B} \Rightarrow \neg(\mathcal{C} \vee \mathcal{D})) \Rightarrow (\exists x)(\mathcal{B} \Rightarrow \neg\mathcal{C} \wedge \neg\mathcal{D})$

d. $\vdash (\forall x)(\exists y)(\mathcal{B} \Rightarrow \mathcal{C}) \Leftrightarrow (\forall x)(\exists y)(\neg\mathcal{B} \vee \mathcal{C})$

e. $\vdash (\forall x)(\mathcal{B} \Rightarrow \neg\mathcal{C}) \Leftrightarrow \neg(\exists x)(\mathcal{B} \wedge \mathcal{C})$

3.14 For each wf $\mathcal{B}$ below, find a wf $\mathcal{C}$ such that $\vdash \mathcal{C} \Leftrightarrow \neg\mathcal{B}$ and negation signs in $\mathcal{C}$ apply ~~ost~~ only to ~~atom~~ atomic wfs.

a. $(\forall x)(\forall y)(\exists z) A_1^3(x,y,z)$

b. $(\forall \varepsilon)(\varepsilon > 0 \Rightarrow (\exists \delta)(\delta > 0 \wedge (\forall x)(|x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon))$

c. $(\forall \varepsilon)(\varepsilon > 0 \Rightarrow (\exists n)(\forall m)(m > n \Rightarrow |a_m - b| < \varepsilon))$

3.15 Let $\mathcal{B}$ be a wf that does not contain $\Rightarrow$ and $\Leftrightarrow$. Exchange universal and existential quantifiers and exchange $\wedge$ and $\vee$. The result $\mathcal{B}^*$ is called the dual of $\mathcal{B}$.

a. In any predicate calculus, prove the following

i. $\vdash \mathcal{B}$ if and only if $\vdash \neg\mathcal{B}^*$

ii $\vdash \mathcal{B} \Rightarrow \mathcal{C}$ if and only if $\vdash \mathcal{C}^* \Rightarrow \mathcal{B}^*$

iii. $\vdash \mathcal{B} \Leftrightarrow \mathcal{C}$ if and only if $\vdash \mathcal{B}^* \Leftrightarrow \mathcal{C}^*$

iv. $\vdash (\exists x)(\mathcal{B} \vee \mathcal{C}) \Leftrightarrow [((\exists x)\mathcal{B}) \vee (\exists x)\mathcal{C}]$. [Hint. Use Exercise 3.9(c), Page-72]

4. Rule C (Choice Rule)

It is very common in mathematics to reason in the following way. Assume that we have proved a wf of the form $(\exists x)\mathcal{B}(x)$. Then we say, let b be an object such that $\mathcal{B}(b)$. We continue the proof, finally arriving at a formula that does not involve the arbitrarily chosen element b.

For example, let us say that we wish to show that $(\exists x)(\mathcal{B}(x) \Rightarrow \mathcal{C}(x))$, $(\forall x)\mathcal{B}(x) \vdash (\exists x)\mathcal{C}(x)$

1.	$(\exists x)(\mathcal{B}(x) \Rightarrow \mathcal{C}(x))$	Hyp
2.	$(\forall x)\mathcal{B}(x)$	Hyp
3.	$\mathcal{B}(b) \Rightarrow \mathcal{C}(b)$ for some b	1
4.	$\mathcal{B}(b)$	2, rule A4
5.	$\mathcal{C}(b)$	3, 4, MP
6.	$(\exists x)\mathcal{C}(x)$	5, rule ~~E4~~ E4

Such a proof seems to be perfectly legitimate on an intuitive basis. Infact, we can achieve the same result without making an arbitrary choice of an element b as in step 3. This can be done as follows:

1. $(\forall x)\mathscr{B}(x)$ Hyp
2. $(\forall x)\neg\mathscr{C}(x)$ Hyp
3. $\mathscr{B}(x)$ 1, rule A4
4. $\neg\mathscr{C}(x)$ 2, rule A4
5. $\neg(\mathscr{B}(x) \Rightarrow \mathscr{C}(x))$ 3, 4, conditional introduction
6. $(\forall x)\neg(\mathscr{B}(x) \Rightarrow \mathscr{C}(x))$ 5, Gen
7. $(\forall x)\mathscr{B}(x), (\forall x)\neg\mathscr{C}(x) \vdash (\forall x)\neg(\mathscr{B}(x) \Rightarrow \mathscr{C}(x))$ 1–6
8. $(\forall x)\mathscr{B}(x) \vdash (\forall x)\neg\mathscr{C}(x) \Rightarrow (\forall x)\neg(\mathscr{B}(x) \Rightarrow \mathscr{C}(x))$ 1-7, Corollary 3.7
9. $(\forall x)\mathscr{B}(x) \vdash \neg(\forall x)\neg(\mathscr{B}(x) \Rightarrow \mathscr{C}(x)) \Rightarrow \neg(\forall x)\neg\mathscr{C}(x)$ 8, contrapositive
10. $(\forall x)\mathscr{B}(x) \vdash (\exists x)(\mathscr{B}(x) \Rightarrow \mathscr{C}(x)) \Rightarrow (\exists x)\mathscr{C}(x)$ Abbreviation of 9
11. $(\exists x)(\mathscr{B}(x) \Rightarrow \mathscr{C}(x)), (\forall x)\mathscr{B}(x) \vdash (\exists x)\mathscr{C}(x)$ 10, MP

In general, any wf that can be proved using a finite number of arbitrary choices can also be proved without such acts of choice. We shall call the rule that permits us to go from $(\exists x)\mathscr{B}(x)$ to $\mathscr{B}(b)$, rule C (or Choice rule). More precisely, a rule C deduction in a first order theory K is defined in the following manner: $\Gamma \vdash_C \mathscr{B}$ if and only if there is a sequence of wfs $\mathscr{D}_1, \ldots, \mathscr{D}_n$ such that $\mathscr{D}_n$ is $\mathscr{B}$ and the following four conditions hold:

1. For each $i < n$, either
   a. $\mathscr{D}_i$ is an axiom of K, or
   b. $\mathscr{D}_i$ is in Γ, or
   c. $\mathscr{D}_i$ follows by MP or Gen from preceding wfs in the sequence, or