17. Without using truth table, prove that $p \vee (p \wedge q) \equiv p$, ($p, q$ are propositional variables) (we write $\mathcal{B} \equiv \mathcal{C}$ if two statement forms $\mathcal{B}$ and $\mathcal{C}$ are logically equivalent)

Solution:

$$
\begin{aligned}
p \vee (p \wedge q) &\equiv (p \wedge T) \vee (p \wedge q) && [\because p \wedge T \equiv p] \\
&\equiv p \wedge (T \vee q) && [\text{Distributive law}] \\
&\equiv p \wedge T && [\because T \vee q \equiv T] \\
&\equiv p && [\because p \wedge T \equiv p]
\end{aligned}
$$

Hence, $p \vee (p \wedge q) \equiv p$

18. Without using truth table, prove that $\sim(p \vee q) \vee (\sim p \wedge q) \equiv \sim p$ where p, q are propositional variables

Solution: Exercise

19. Without using truth table, prove that $p \to (q \to r) \equiv (p \wedge q) \to r$, where p, q, r are propositions.

Solution:

$$
\begin{aligned}
p \to (q \to r) &\equiv p \to (\sim q \vee r) && (\text{Conditional equivalence, i.e., } p \to q \equiv \sim p \vee q) \\
&\equiv \sim p \vee (\sim q \vee r) && (\text{Conditional equivalence}) \\
&\equiv (\sim p \vee \sim q) \vee r && (\text{Associative Law}) \\
&\equiv \sim(p \wedge q) \vee r && (\text{De Morgan's Law}) \\
&\equiv (p \wedge q) \to r && (\text{Conditional equivalence})
\end{aligned}
$$

Hence, $p \to (q \to r) \equiv (p \wedge q) \to r$

20. Without using truth table prove that $\sim p \wedge (\sim q \wedge r) \vee (q \wedge r) \vee (p \wedge r) \equiv r$

Solution: $\sim p \wedge (\sim q \wedge r) \vee (q \wedge r) \vee (p \wedge r)$

$$
\begin{aligned}
&\equiv ((\sim p \wedge \sim q) \wedge r) \vee ((q \wedge r) \vee (p \wedge r)) && (\text{Associative Law}) \\
&\equiv ((\sim p \wedge \sim q) \wedge r) \vee ((q \vee p) \wedge r)) && (\text{Distributive Law}) \\
&\equiv ((\sim p \wedge \sim q) \vee (q \vee p)) \wedge r && (\text{Distributive Law}) \\
&\equiv (\sim(p \vee q) \vee (p \vee q)) \wedge r && (\text{De Morgan's Law and Commutative Law}) \\
&\equiv T \wedge r && (\text{Negation Law}) \\
&\equiv r && (\text{Identity Law})
\end{aligned}
$$

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Hence, $\sim p \wedge (\sim q \wedge r) \vee (q \wedge r) \vee (p \wedge r) \equiv r$

20. Without using truth table, prove that $(p \wedge (\sim p \vee q)) \vee (q \wedge \sim (p \wedge q)) \equiv q$, where p, q are propositional variables.

Solution

$(p \wedge (\sim p \vee q)) \vee (q \wedge \sim (p \wedge q))$

$\equiv ((p \wedge \sim p) \vee (p \wedge q)) \vee (q \wedge \sim (p \wedge q))$ (Distributive Law)

$\equiv (F \vee (p \wedge q)) \vee (q \wedge (\sim p \vee \sim q))$ (Negation Law, De Morgan's law)

$\equiv (p \wedge q) \vee ((q \wedge \sim p) \vee (q \wedge \sim q))$ (Distributive Law & Identity law)

$\equiv (p \wedge q) \vee ((q \wedge \sim p) \vee F)$ (Negation Law)

$\equiv (p \wedge q) \vee (q \wedge \sim p)$ (Identity law)

$\equiv (q \wedge p) \vee (q \wedge \sim p)$ (Commutative law)

$\equiv q \wedge (p \vee \sim p)$ (Distributive Law)

$\equiv q \wedge T$ (Negation ~~law~~ law)

$\equiv q$ (Identity Law)

Hence $(p \wedge (\sim p \vee q)) \vee (q \wedge \sim (p \wedge q)) \equiv q$

21. Without using truth table, prove that $(\sim p \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \equiv r$, where p, q, r are propositional variables

Solution: Exercise

22. Show that $P \Rightarrow (Q \Rightarrow R) \equiv P \Rightarrow (\neg Q \vee R) \equiv (P \wedge Q) \Rightarrow R$ without using truth table.

Solution: $P \Rightarrow (Q \Rightarrow R) \equiv P \Rightarrow (\neg Q \vee R)$ (Conditional equivalence)

$\equiv \neg P \vee (\neg Q \vee R)$ (conditional equivalence)

$\equiv (\neg P \vee \neg Q) \vee R$ (Associative Law)

$\equiv \neg (P \wedge Q) \vee R$ (De Morgan's Law)

$\equiv (P \wedge Q) \Rightarrow R$ (Conditional equivalence)

Hence, $P \Rightarrow (Q \Rightarrow R) \equiv P \Rightarrow (\neg Q \vee R) \equiv (P \wedge Q) \Rightarrow R$

23. Without using truth table prove the following

(i) $P \Rightarrow (Q \Rightarrow P) \equiv \neg P \Rightarrow (P \Rightarrow Q)$

(ii) $P \Rightarrow (Q \vee R) \equiv (P \Rightarrow Q) \vee (P \Rightarrow R)$

Solution : Exercises

24. Without using truth table prove that followings are tautologies.

(i) $((p \vee \sim q) \wedge (\sim p \vee \sim q)) \vee q$

(ii) $((p \vee q) \wedge \sim(\sim p \wedge (\sim q \vee \sim r))) \vee (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)$

Solution :

(i) $((p \vee \sim q) \wedge (\sim p \vee \sim q)) \vee q$

$\equiv ((p \wedge \sim p) \vee \sim q) \vee q$ (Distributive Law)

$\equiv (F \vee \sim q) \vee q$ (Negation Law)

$\equiv \sim q \vee q$ (~~Iditity Law~~) (Identity law)

$\equiv T$ (~~Iditi~~ Identity law)

So, $((p \vee \sim q) \wedge (\sim p \vee \sim q)) \vee q$ is a tautology

(ii) $((p \vee q) \wedge \sim(\sim p \wedge (\sim q \vee \sim r))) \vee (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)$

$\equiv ((p \vee q) \wedge \sim(\sim p \wedge \sim(q \wedge r))) \vee \sim(p \vee q) \vee \sim(p \vee r)$ (De Morgan's law)

$\equiv ((p \vee q) \wedge \sim(\sim(p \vee (q \wedge r)))) \vee \sim(p \vee q) \vee \sim(p \vee r)$ (De Morgan's Law)

$\equiv ((p \vee q) \wedge (p \vee (q \wedge r))) \vee \sim(p \vee q) \vee \sim(p \vee r)$ (Double negation Law)

$\equiv ((p \vee q) \wedge ((p \vee q) \wedge (p \wedge r))) \vee \sim(p \vee q) \vee \sim(p \vee r)$ (Distributive Law)

$\equiv (((p \vee q) \wedge (p \vee q)) \wedge (p \wedge r)) \vee \sim(p \vee q) \vee \sim(p \vee r)$ (Associative Law)

$\equiv ((p \vee q) \wedge (p \wedge r)) \vee \sim((p \vee q) \wedge (p \vee r))$ (Idempotent Law and De Morgan's Law)

$\equiv T$ (Negation Law)

So, $((p \vee q) \wedge \sim(\sim p \wedge (\sim q \vee \sim r))) \vee (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)$ is a tautology.

About proof

Without introduction of axiomatic theory in propositional logic, we define something about proofs using mathematical logic as follows:

Let p, q be propositions. If $p \to q$ is a tautology we say that p logically implies q or q logically follows from p

Let $p_1, p_2, \ldots, p_n$ and q be propositions such that $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \to q$ is a tautology, then we say that q logically follows from $p_1, p_2, \ldots, p_n$ and in this case, we write

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline q \end{array}$$

The statements $p_1, p_2, \ldots, p_n$ are called premises or hypothesis

The statement q is called the conclusion

The premises $p_1, \ldots p_n$ and the conclusion q as a whole is called an argument.

An argument is called logically valid if and only if conjunction of all premises logically implies the conclusion or equivalently, if the premises are all true, the conclusion must also be true.

A theorem is a proposition that can be proved to be true and hence to prove theorem means to show that implication of the form $(p_1 \wedge p_2 \cdots, \wedge p_n) \to q$ is a tautology.

<u>Rules of Inference</u> : The main aim of logic is to provide rules of inference to get a conclusion from some ~~premises~~ premises. If a conclusion is drawn from a set of premises by using the accepted rules of reasoning then it is called a deduction or a formal proof. Now, we describe the process of derivation in which three rules, namely, Rule P, Rule T and Rule CP are used, by which we can check the validity of the argument

Rule P : At any point of derivation, we may introduce a premise.

Rule T : In the derivation, we may introduce a formula S if S is ~~tautology~~ tautologically implied by any one or more of the preceding formulae.

Rule CP: If we can derive s from r and a set of premises then we can derive $r \to s$ from the set of premises only.

The rule CP is generally used if the ~~conclus~~ conclusion is of the form: $r \to s$.

The rules of inference which are often used for checking the validity of an argument are listed below

1. Modus Ponens

$$\begin{array}{c} p \to q \\ p \\ \hline q \end{array} \qquad \text{OR, } ((p\to q)\wedge q) \to q \text{ is a tautology}$$

2. Modus Tollens:

$$\begin{array}{c} p \to q \\ \sim q \\ \hline \sim p \end{array} \qquad \text{OR, } ((p\to q)\wedge(\sim q)) \to \sim p \text{ is a tautology}$$

3. Hypothetical Syllogism:

$$\begin{array}{c} p \to q \\ q \to r \\ \hline p \to r \end{array} \qquad \text{OR, } ((p\to q)\wedge(q\to r)) \to (p \to r) \text{ is a tautology}$$

4. Disjunctive Syllogism:

$$\begin{array}{c} p \vee q \\ \sim q \\ \hline p \end{array} \qquad \text{OR, } ((p\vee q)\wedge(\sim q)) \to p \text{ is a tautology}$$

$$\begin{array}{c} p \vee q \\ \sim p \\ \hline q \end{array} \qquad \text{OR, } ((p\vee q)\wedge(\sim p)) \to q \text{ is a tautology}$$

5. Addition:

$$\begin{array}{c} p \\ \hline p \vee q \end{array} \qquad \text{OR, } p \to p \vee q \text{ is a tautology}$$

$$\begin{array}{c} q \\ \hline p \vee q \end{array} \qquad \text{OR, } q \to p \vee q \text{ is a tautology}$$