d. there is a preceeding wf $(\exists x)\mathscr{C}(x)$ such that $\mathscr{D}_i$ is $\mathscr{C}(d)$, where d is a new individual constant (rule C)

2. As axioms in condition 1(a), we also can use all logical axioms that involve the new individual constants already introduced in the sequence by applications of rule C.

3. No application of Gen is made using a variable that is free in some $(\exists x)\mathscr{C}(x)$ to which rule C has been previously applied.

4. $\mathscr{B}$ contains none of the new individual constants introduced in the sequence in any application of rule C.

A word should be said about the reason for including condition 3. If an application of rule C to a wf $(\exists x)\mathscr{C}(x)$ ~~to~~ yields $\mathscr{C}(d)$, then the object referred to by d may depend on the values of the free variables in $(\exists x)\mathscr{C}(x)$. So that one of object may not satisfy $\mathscr{C}(x)$ for all values of the free variables in $(\exists x)\mathscr{C}(x)$. For example, without clause 3, we could proceed as follows:

1. $(\forall x)(\exists y)A_1^2(x,y)$ — Hyp
2. $(\exists y)A_1^2(x,y)$ — 1, rule A4
3. $A_1^2(x,d)$ — 2, rule C
4. $(\forall x)A_1^2(x,d)$ — 3, Gen
5. $(\exists y)(\forall x)A_1^2(x,y)$ — 4, rule E4

However, there is an interpretation for which $(\forall x)(\exists y)A_1^2(x,y)$ is true but $(\exists y)(\forall x)A_1^2(x,y)$ is false. Take the domain to be the set of integers and let $A_1^2(x,y)$ means ~~that~~ $x<y$.

Proposition 4.1 If $\Gamma \vdash_C \mathscr{B}$, then $\Gamma \vdash \mathscr{B}$. Moreover, from the following proof it is easy to verify that, if there is an application of Gen in the new proof of $\mathscr{B}$ from Γ using a certain variable and applied to a wf depending upon a certain wf of Γ, then there was such an application of Gen in the original proof.

Proof: Let $(\exists y_1)\mathscr{C}_1(y_1), \ldots, (\exists y_k)\mathscr{C}_k(y_k)$ be the wfs in order of occurrence to which rule C is applied in the proof $\Gamma \vdash_C \mathscr{B}$, and let $d_1, d_2, \ldots, d_k$ be the corresponding new individual constants. Then $\Gamma, \mathscr{C}_1(d_1), \ldots, \mathscr{C}_k(d_k) \vdash \mathscr{B}$. Now, by condition 3 of the definition above, Corollary 3.7 is applicable, yielding $\Gamma, \mathscr{C}_1(d_1), \ldots, \mathscr{C}_{k-1}(d_{k-1}) \vdash \mathscr{C}_k(d_k) \Rightarrow \mathscr{B}$. We replace d_k everywhere by a variable z that does not occur in the proof. Then

$$\Gamma, \mathscr{C}_1(d_1), \ldots, \mathscr{C}_{k-1}(d_{k-1}) \vdash \mathscr{C}_k(z) \Rightarrow \mathscr{B}.$$

and, by Gen,

$$\Gamma, \mathscr{C}_1(d_1), \ldots, \mathscr{C}_{k-1}(d_{k-1}) \vdash (\forall z)(\mathscr{C}_k(z) \Rightarrow \mathscr{B})$$

Hence, by Exercise 3.12(d) (page-75),

$$\Gamma, \mathscr{C}_1(d_1), \ldots, \mathscr{C}_{k-1}(d_{k-1}) \vdash (\exists y_k)\, \mathscr{C}_k(y_k) \Rightarrow \mathscr{B}$$

But,

$$\Gamma, \mathscr{C}_1(d_1), \ldots, \mathscr{C}_{k-1}(d_{k-1}) \vdash (\exists y_k)\, \mathscr{C}_k(y_k)$$

Hence, by MP,

$$\Gamma, \mathscr{C}_1(d_1), \ldots, \mathscr{C}_{k-1}(d_{k-1}) \vdash \mathscr{B}$$

Repeating this argument, we can eliminate $\mathscr{C}_{k-1}(d_{k-1}), \ldots, \mathscr{C}_1(d_1)$, one after the other, finally obtaining $\Gamma \vdash \mathscr{B}$.

Example $\vdash (\forall x)(\mathscr{B}(x) \Rightarrow \mathscr{C}(x)) \Rightarrow ((\exists x)\, \mathscr{B}(x) \Rightarrow (\exists x)\mathscr{C}(x))$

1. $(\forall x)(\mathscr{B}(x) \Rightarrow \mathscr{C}(x))$	Hyp
2. $(\exists x)\mathscr{B}(x)$	Hyp
3. $\mathscr{B}(d)$	2, rule C
4. $\mathscr{B}(d) \Rightarrow \mathscr{C}(d)$	1, rule A4
5. $\mathscr{C}(d)$	3,4, MP
6. $(\exists x)\, \mathscr{C}(x)$	5, rule E4
7. $(\forall x)(\mathscr{B}(x) \Rightarrow \mathscr{C}(x)), (\exists x)\mathscr{B}(x) \vdash_C (\exists x)\mathscr{C}(x)$	1-6
8. $(\forall x)(\mathscr{B}(x) \Rightarrow \mathscr{C}(x)), (\exists x)\mathscr{B}(x) \vdash (\exists x)\mathscr{C}(x)$	7, Proposition 4.1
9. $(\forall x)(\mathscr{B}(x) \Rightarrow \mathscr{C}(x)) \vdash (\exists x)\mathscr{B}(x) \Rightarrow (\exists x)\mathscr{C}(x)$	1-8, Corollary 3.7
10. $\vdash (\forall x)(\mathscr{B}(x) \Rightarrow \mathscr{C}(x)) \Rightarrow ((\exists x)\mathscr{B}(x) \Rightarrow (\exists x)\mathscr{C}(x))$	1-9, Corollary 3.7

Exercises

Use rule C and Proposition ~~4~~. 4.1 to prove 3.16 – 3.23

3.16 $\vdash (\exists x)(\mathcal{B}(x) \Rightarrow \mathcal{C}(x)) \Rightarrow ((\forall x)\mathcal{B}(x) \Rightarrow (\exists x)\mathcal{C}(x))$

3.17 $\vdash \neg(\exists y)(\forall x)(A_1^2(x,y) \Leftrightarrow \neg A_1^2(x,x))$

3.18 $\vdash [(\forall x)(A_1^1(x) \Rightarrow A_2^1(x) \vee A_3^1(x)) \wedge \neg(\forall x)(A_1^1(x) \Rightarrow A_2^1(x))] \Rightarrow (\exists x)(A_1^1(x) \wedge A_3^1(x))$

3.19 $\vdash [(\exists x)\mathcal{B}(x)] \wedge [(\forall x)\mathcal{C}(x)] \Rightarrow (\exists x)(\mathcal{B}(x) \wedge \mathcal{C}(x))$

3.20 $\vdash (\exists x)\mathcal{C}(x) \Rightarrow (\exists x)(\mathcal{B}(x) \vee \mathcal{C}(x))$

3.21 $\vdash (\exists x)(\exists y)\mathcal{B}(x,y) \Leftrightarrow (\exists y)(\exists x)\mathcal{B}(x,y)$

3.22 $\vdash (\exists x)(\forall y)\mathcal{B}(x,y) \Rightarrow (\forall y)(\exists x)\mathcal{B}(x,y)$

3.23 $\vdash (\exists x)(\mathcal{B}(x) \wedge \mathcal{C}(x)) \Rightarrow ((\exists x)\mathcal{B}(x) \wedge (\exists x)\mathcal{C}(x))$

we state ~~two~~ three Theorems without proof

~~Proposition~~ ~~(Soundness Theorem)~~ Proposition 4.2 (Soundness Theorem) If a wf $\mathcal{B}$ is a theorem of a theory ~~$\mathcal{K}$~~ K then it is logically valid in K.

Proposition 4.3 (Completeness Theorem) In any predicate calculus, if a wf $\mathcal{B}$ is logically valid then it is a theorem.

Proposition 4.4 (Compactness Theorem) If all finite finite subsets of the set of axioms of a theory K have models, then K has a model.

5. Prenex Normal Forms

A wf $(Q_1 y_1)\cdots(Q_n y_n)\mathcal{B}$, where each $(Q_i y_i)$ is either $(\forall y_i)$ or $(\exists y_i)$, y_i is different from y_j for $i \neq j$ and $\mathcal{B}$ contains no quantifiers, is said to be in prenex normal form. (We include the case $n=0$, when there are no quantifiers at all.)

If x_i and x_j are distinct, then $\mathcal{B}(x_i)$ and $\mathcal{B}(x_j)$ are said to be similar if and only if x_j is free for x_i in $\mathcal{B}(x_i)$ and $\mathcal{B}(x_i)$ has no free occurences of x_j. It is assumed here that $\mathcal{B}(x_j)$ ~~arr~~ arises from $\mathcal{B}(x_i)$ by substituting x_j for all free occurences of x_i. It is easy to see that, if $\mathcal{B}(x_i)$ and $\mathcal{B}(x_j)$ are similar, then x_i is free for x_j in $\mathcal{B}(x_j)$ and $\mathcal{B}(x_j)$ has no free occurences of x_i. Thus, if $\mathcal{B}(x_i)$ and $\mathcal{B}(x_j)$ are similar, then $\mathcal{B}(x_j)$ and $\mathcal{B}(x_i)$ are similar. Intuitively, $\mathcal{B}(x_i)$ and $\mathcal{B}(x_j)$ are similar if and only if $\mathcal{B}(x_i)$ and $\mathcal{B}(x_j)$ are the same except that $\mathcal{B}(x_i)$ has free occurences of x_i in exactly those places where $\mathcal{B}(x_j)$ has free occurences of x_j.

Example
$(\forall x_3)\left[A_1^2(x_1,x_3) \vee A_1^1(x_1)\right]$ and $(\forall x_3)\left[A_1^2(x_2,x_3) \vee A_1^1(x_2)\right]$ are similar

We state two lemmas (without proof)

Lemma 5.1 If $\mathcal{B}(x_i)$ and $\mathcal{B}(x_j)$ are similar, then $\vdash (\forall x_i)\mathcal{B}(x_i) \Leftrightarrow (\forall x_j)\mathcal{B}(x_j)$

Lemma 5.2 In any theory, if y is not free in $\mathcal{D}$, and $\mathcal{C}(x)$ and $\mathcal{C}(y)$ are similar, then the following hold.

a. $\vdash ((\forall x)\mathcal{C}(x) \Rightarrow \mathcal{D}) \Leftrightarrow (\exists y)(\mathcal{C}(y) \Rightarrow \mathcal{D})$
b. $\vdash ((\exists x)\mathcal{C}(x) \Rightarrow \mathcal{D}) \Leftrightarrow (\forall y)(\mathcal{C}(y) \Rightarrow \mathcal{D})$
c. $\vdash (\mathcal{D} \Rightarrow (\forall x)\mathcal{C}(x)) \Leftrightarrow (\forall y)(\mathcal{D} \Rightarrow \mathcal{C}(y))$
d. $\vdash (\mathcal{D} \Rightarrow (\exists x)\mathcal{C}(x)) \Leftrightarrow (\exists y)(\mathcal{D} \Rightarrow \mathcal{C}(y))$
e. $\vdash \neg(\forall x)\mathcal{C} \Leftrightarrow (\exists x)\neg\mathcal{C}$
f. $\vdash \neg(\exists x)\mathcal{C} \Leftrightarrow (\forall x)\neg\mathcal{C}$

We can prove that, for every wf, we can ~~contstruct~~ construct an equivalent prenex normal form. That is, ~~there is an~~ we can ~~effecting~~ prove that there is an effective procedure for transforming any wf $\mathcal{B}$ into a wf $\mathcal{C}$ in prenex normal form such that $\vdash \mathcal{B} \Leftrightarrow \mathcal{C}$.

Examples
1. Let $\mathcal{B}$ be $(\forall x)\left(A_1^1(x) \Rightarrow (\forall y)\left(A_2^2(x,y) \Rightarrow \neg(\forall z)(A_3^2(y,z))\right)\right)$

By part (e) of Lemma 5.2: $(\forall x)\left(A_1^1(x) \Rightarrow (\forall y)\left[A_2^2(x,y) \Rightarrow (\exists z)\neg A_3^2(y,z)\right]\right)$

By part (d): $(\forall x)(A_1^1(x) \Rightarrow (\forall y)(\exists u)[A_2^2(x,y) \Rightarrow \neg A_3^2(y,u)])$

By part (c): $(\forall x)(\forall v)(A_1^1(x) \Rightarrow (\exists u)[A_2^2(x,v) \Rightarrow \neg A_3^2(v,u)])$

By part (d): $(\forall x)(\forall v)(\exists w)(A_1^1(x) \Rightarrow (A_2^2(x,v) \Rightarrow \neg A_3^2(v,w)])$

Changing bound variables: $(\forall x)(\forall y)(\exists z)(A_1^1(x) \Rightarrow (A_2^2(x,y) \Rightarrow \neg A_3^2(y,z)))$

which is the equivalent prenex normal form

2. Let $\mathcal{B}$ be $A_1^2(x,y) \Rightarrow (\exists y)[A_1^1(y) \Rightarrow ([(\exists x)A_1^1(x)] \Rightarrow A_2^1(y))]$

By part (b): $A_1^2(x,y) \Rightarrow (\exists y)[A_1^1(y) \Rightarrow (\forall u)[A_1^1(u) \Rightarrow A_2^1(y)]])$

By part (c): $A_1^2(x,y) \Rightarrow (\exists y)(\forall v)[A_1^1(y) \Rightarrow [A_1^1(v) \Rightarrow A_2^1(y)]])$

By part (d): $(\exists w)(A_1^2(x,y) \Rightarrow (\forall v)(A_1^1(w) \Rightarrow [A_1^1(v) \Rightarrow A_2^1(w)]])$

By part (c): $(\exists w)(\forall z)(A_1^2(x,y) \Rightarrow (A_1^1(w) \Rightarrow [A_1^1(z) \Rightarrow A_2^1(w)]])$

which is the equivalent prenex normal form.

Exercise

3.24 Find the prenex normal forms equivalent to the following wfs.

a. $[(\forall x)(A_1^1(x) \Rightarrow A_1^2(x,y))] \Rightarrow ([(\exists y)A_1^1(y)] \Rightarrow (\exists z)A_1^2(y,z))$

b. $(\exists x)A_1^2(x,y) \Rightarrow (A_1^1(x) \Rightarrow \neg(\exists u)A_1^2(x,u))$

6. First order Theories with equality

Let K be a theory that has one of its predicate letters A_1^2. Let us write $t = s$ as an abbreviation for $A_1^2(t,s)$ and $t \neq s$ for $\neg A_1^2(t,s)$. Then K is called a first order theory with equality (or simply a theory with equality) if the following are theorems of K:

(A6) $(\forall x_1)\, x_1 = x_1$ (reflexivity of equality)

(A7) $x = y \Rightarrow (\mathcal{B}(x,x) \Rightarrow \mathcal{B}(x,y))$ (substitutivity of equality)

where x, and y are any variables, $\mathcal{B}(x,x)$ is any wf, and $\mathcal{B}(x,y)$ arises from $\mathcal{B}(x,x)$ by replacing some, but not necessarily all, free occurences of x and y, with the provision that y is free