for x in $\mathcal{B}(x,x)$. Thus, $\mathcal{B}(x,y)$ may or may not contain free occurences of x.

The numbering (A6) and (A7) is a continuation of the numbering of the logical axioms.

Proposition 6.1 In any theory with equality,

a. $\vdash t=t$ for any term t

b. $\vdash t=s \Rightarrow s=t$ for any terms t and s

c. $\vdash t=s \Rightarrow (s=r \Rightarrow t=r)$ for any terms t, s, and r.

Proof: a. By (A6), $\vdash (\forall x_1)x_1=x_1$, Hence by rule A4 $\vdash t=t$

b. Let x and y be variables not occuring in t or s. Letting $\mathcal{B}(x,x)$ be $x=x$ and $\mathcal{B}(x,y)$ be $y=x$ in schema (A7), $\vdash x=y \Rightarrow (x=x \Rightarrow y=x)$, But, by (a), $\vdash x=x$, So, by an instance of a tautology $(A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))$ and two applications of MP, we have $\vdash x=y \Rightarrow y=x$. Two application of Gen yield $\vdash (\forall x)(\forall y)(x=y \Rightarrow y=x)$, and then two applications of rule A4 give $\vdash t=s \Rightarrow s=t$

c. Let x, y, and z be three variables not occuring in t, s or r. Letting $\mathcal{B}(y,y)$ be $y=z$ and $\mathcal{B}(y,x)$ be $x=z$ in (A7), with x and y interchanged, we obtain $\vdash y=x \Rightarrow (y=z \Rightarrow x=z)$, But, by (b), $\vdash (x=y) \Rightarrow y=x$.

Hence using an instance of the tautology $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$ and two applications of MP, we obtain, $\vdash x=y \Rightarrow (y=z \Rightarrow x=z)$. By three application of Gen, $\vdash (\forall x)(\forall y)(\forall z)(x=y \Rightarrow (y=z \Rightarrow x=z))$ and then by three uses of rule A4, $\vdash t=s \Rightarrow (s=r \Rightarrow t=r)$.

Exercises Prove the following in any theory with equality

a. $\vdash (\forall x)(\mathcal{B}(x) \Leftrightarrow (\exists y)(x=y \wedge \mathcal{B}(y)))$ if y does not occur in $\mathcal{B}(x)$

b. $\vdash (\forall x)(\mathcal{B}(x) \Leftrightarrow (\forall y)(x=y \Rightarrow \mathcal{B}(y)))$ if y does not occur in $\mathcal{B}(x)$

c. $\vdash (\forall x)(\exists y)\, x=y$

d. $\vdash x=y \Rightarrow f(x)=f(y)$, where f is any function letter of one argument

e. $\vdash \mathcal{B}(x) \wedge x=y \Rightarrow \mathcal{B}(y)$, if y is free for x in $\mathcal{B}(x)$

f. $\vdash \mathcal{B}(x) \wedge \neg\mathcal{B}(y) \Rightarrow x \neq y$, if y is free for x in $\mathcal{B}(x)$

Some examples of First order theories :

1. First order theory ~~of~~ ~~of groups~~ G of groups: Here the predicate letter is $=$, function letter f_1^2, and individual constant a_1. We abbreviate $f_1^2(t,s)$ by $t+s$ and a_1 by 0.

The proper axioms are the following:

a. $x_1+(x_2+x_3) = (x_1+x_2)+x_3$

b. $x_1+0 = x_1$

c. $(\forall x_1)(\exists x_2)\, x_1+x_2 = 0$

d. $x_1 = x_1$

e. $x_1 = x_2 \Rightarrow x_2 = x_1$

f. $x_1 = x_2 \Rightarrow (x_2 = x_3 \Rightarrow x_1 = x_3)$

g. $x_1 = x_2 \Rightarrow (x_1+x_3 = x_2+x_3 \wedge x_3+x_1 = x_3+x_2)$

This G is a first order theory with equality (we can prove it).

If one add to the axioms the following wf:

h. $x_1+x_2 = x_2+x_1$

the new theory is called elementary (or first order) theory of abelian groups.

(Note: we sometimes use 'elementary' instead of first order)

2. Elementary theory of ~~rings~~ R of rings: Here the predicate letter is $=$, two function letters are f_1^2 and f_2^2 and individual constants are a_1 and a_2. Abbreviate $f_1^2(t,s)$ by $t+s$ and $f_2^2(t,s)$ by $t \cdot s$, and a_1 and a_2 by 0 and 1. As proper axioms, we take (a) – (h) of Example 1 plus the following:

i. $x_1 = x_2 \Rightarrow (x_1 \cdot x_3 = x_2 \cdot x_3 \wedge x_3 \cdot x_1 = x_3 \cdot x_2)$

j. $x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3$

k. $x_1 \cdot (x_2+x_3) = (x_1 \cdot x_2)+(x_1 \cdot x_3)$

l. $(x_2+x_3) \cdot x_1 = (x_2 \cdot x_1)+(x_3 \cdot x_1)$

Then R is a first order theory with equality (we can prove it). This theory R is called elementary theory of rings. If we add the following axiom:

m. $x_1 \cdot x_2 = x_2 \cdot x_1$, Then is called Theory of commutative ring. (Here axiom k and l coincide). If we add three more axioms

n. $x_1 \cdot 1 = x_1$

o. $x_1 \neq 0 \Rightarrow (\exists x_2)\, x_1 \cdot x_2 = 1$

p. $0 \neq 1$

Then it is also a theory of equality and it is called the elementary Theory of Fields.

—————— X ——————