n. $x_1 \cdot 1 = x_1$

o. $x_1 \neq 0 \Rightarrow (\exists x_2)\, x_1 \cdot x_2 = 1$

p. $0 \neq 1$

Then it is also a theory of equality and it is called the elementary Theory of Fields.

XVI

Some Problems with answers

1. Let $\oplus$ denotes the binary operation exclusive or. Thus $A \oplus B$ stands for A or B but not both. Write the truth table for $\oplus$.

Ans: The truth table for $A \oplus B$ is

A	B	$A \oplus B$
T	T	F
F	T	T
T	F	T
F	F	F

2. Construct the truth table for the statement form $(p \to q) \vee (\sim p)$ and $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$

Ans: The truth table for $(p \to q) \vee (\sim p)$ is

p	q	$\sim p$	$p \to q$	$(p \to q) \vee (\sim p)$
T	T	F	T	T
F	T	T	T	T
T	F	F	F	F
F	F	T	T	T

The truth table for $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$ is

p	q	r	$p \to q$	$p \to r$	$q \to r$	$p \to (q \to r)$	$(p \to q) \to (p \to r)$	$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
T	T	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T	T
T	F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
F	T	F	T	T	F	T	T	T
T	F	F	F	F	T	T	T	T
F	F	F	T	T	T	T	T	T

3. Construct the truth table for $(p \to q) \to (\sim q \to \sim p)$

Ans: The required truth table is

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$	$(p \to q) \to (\sim q \to \sim p)$
T	T	F	F	T	T	T
F	T	T	F	T	T	T
T	F	F	T	F	F	T
F	F	T	T	T	T	T

4. Construct the truth table for $((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$

Ans: The required truth table is

A	B	C	$A \Rightarrow B$	$A \Rightarrow C$	$B \Rightarrow C$	$(A \Rightarrow B) \wedge (B \Rightarrow C)$	$((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$
T	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T
T	F	T	F	T	T	F	T
F	F	T	T	T	T	T	T
T	T	F	T	F	F	F	T
F	T	F	T	T	F	F	T
T	F	F	F	F	T	F	T
F	F	F	T	T	T	T	T

5. Using the statements p: Ram is tall and q: Ram is happy, write the following statements in symbolic form:
(i) Ram is short but happy (ii) Ram is neither tall nor happy
(iii) Ram is tall or unhappy (iv) Ram is short or he is both tall and unhappy

Ans: The symbolic forms are: (i) $(\sim p) \wedge q$ (ii) $(\sim p) \wedge (\sim q)$
(iii) $p \vee (\sim q)$ (iv) $(\sim p) \vee (p \wedge (\sim q))$

6. Write the following sentences as statement forms, using statement letters to stand for the atomic sentences – that is, those sentences that are not built up out of other sentences:

(a) If Mr. Ray is happy, Mrs. Ray is not happy, and if Mr. Ray is not happy, Mrs. Ray is not happy

(b) A sufficient condition for x to be odd is that x is prime.

(c) A necessary condition for a sequence s to converge is that s be bounded

(d) A necessary an sufficient condition for Anwar to be happy is that he has money, a house and peace in mind.

(e) Nabin goes to the movies only if a comedy is playing.

(f) The bribe will be paid if and only if the goods are delivered.

(g) If x is positive, x^2 is positive

(h) Karpov will win the chess tournament unless Kasparov wins today.

Ans: (a) Let A: Mr. Ray is happy and B: Mrs. Ray is happy. Then the statement form is $(A \Rightarrow (\neg B)) \wedge ((\neg A) \Rightarrow B)$

(b) Let A: x is prime and B: x is odd. The statement form is

$A \Rightarrow B$

(c) Let A: the sequence s converges, B: the sequence s is bounded

The statement form is $A \Rightarrow B$

(d) Let A: Anwar is happy, B: Anwar has money,

C: Anwar has a house D: Anwar has peace in mind.

The statement form is $A \Leftrightarrow (B \wedge (C \wedge D))$

(e) Let A: Nabin goes to the movies and B: A comedy is playing.

The statement form is $A \Rightarrow B$

(f) Let A: The bribe will be paid and B: the goods are delivered

The statement form is $A \Leftrightarrow B$

(g) Let p: x is positive and q: x^2 is positive. The statement form is $p \rightarrow q$

(h) Let A: Kasparov wins today and B: Karpov will win the tournament.

The statement form is $\neg A \Rightarrow B$

7. Let the two propositions be A: It is cold and B: It is raining. Write the statement ~~against~~ against the following symbol:

(i) $B \wedge \neg A$ (ii) $A \vee B$ (iii) $\neg (A \vee B)$

Ans: (i) It is raining but not cold.

(ii) Either it is cold or it is raining

(iii) ~~Neither it is raining nor it is~~
Neither it is cold nor it is raining.

8. Determine whether the followings are tautologies:

(a) $(((A \Rightarrow B) \Rightarrow B) \Rightarrow B)$

(b) $(((p \to q) \to q) \to p)$

(c) $(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)$

(d) $(((B \Rightarrow C) \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B))$

(e) $((p \vee (\sim(q \wedge r))) \to ((p \leftrightarrow r) \vee q))$

(f) $(p \to (q \to (q \to p)))$

(g) $((p \wedge q) \to (p \vee r))$

(h) $((p \leftrightarrow q) \leftrightarrow (p \leftrightarrow (q \leftrightarrow p)))$

(i) $((p \to q) \vee (q \to p))$

(j) $((\neg(A \Rightarrow B)) \Rightarrow A)$

Solution: (a)

A	B	$A \Rightarrow B$	$(A \Rightarrow B) \Rightarrow B$	$((A \Rightarrow B) \Rightarrow B) \Rightarrow B$
T	T	T	T	T
F	T	T	T	T
T	F	F	T	F
F	F	T	F	T

So, there is a F in the last column. So, $(((A \Rightarrow B) \Rightarrow B) \Rightarrow B)$ is not a tautology

(b), (c), (d) → Exercises

(e)

p	q	r	$\sim(q \wedge r)$	$p \leftrightarrow r$	$(p \vee (\sim(q \wedge r)))$	$(p \leftrightarrow r) \vee q$	$((p \vee (\sim(q \wedge r)))) \to ((p \leftrightarrow r) \vee q))$
T	T	T	F	T	T	T	T
F	T	T	F	F	F	T	T
T	F	T	T	T	T	T	T
F	F	T	T	F	T	F	F
T	T	F	T	F	T	T	T
F	T	F	T	T	T	T	T
T	F	F	T	F	T	F	F
F	F	F	T	T	T	T	T

As there is F in the last column, the statement form is not a tautology

(f) → exercise

(g)

p	q	r	$p \wedge q$	$p \vee r$	$(p \wedge q) \to (p \vee r)$
T	T	T	T	T	T
F	T	T	F	T	T
T	F	T	F	T	T
F	F	T	F	T	T
T	T	F	T	T	T
F	T	F	F	F	T
T	F	F	F	T	T
F	F	F	F	F	T

So, from the last column, we see that $((p \wedge q) \to (p \vee r))$ is a tautology.

(h), (i), (j) → Exercises.

9. Determine whether the following pairs are logically equivalent.

(a) $((A \Rightarrow B) \Rightarrow A)$ and A

(b) $(A \Leftrightarrow B)$ and $((A \Rightarrow B) \wedge (B \Rightarrow A))$

(c) $((\neg A) \vee B)$ and $((\neg B) \vee A)$

(d) $(\neg (A \Leftrightarrow B))$ and $(A \Leftrightarrow (\neg B))$

(e) $(p \vee (q \leftrightarrow r))$ and $((p \vee q) \leftrightarrow (p \vee r))$

(f) $(p \to (q \leftrightarrow r))$ and $((p \to q) \leftrightarrow (p \to r))$

(g) $(p \wedge (q \leftrightarrow r))$ and $((p \wedge q) \leftrightarrow (p \wedge r))$

Solution: (a) we know that two statement forms $\mathcal{B}$ and $\mathcal{C}$ are logically equivalent if and only if $(\mathcal{B} \Leftrightarrow \mathcal{C})$ is a tautology. So we use it.

A	B	$A \Rightarrow B$	$((A \Rightarrow B) \Rightarrow A)$	$((A \Rightarrow B) \Rightarrow A) \Leftrightarrow A$
T	T	T	T	T
F	T	T	F	T
T	F	F	T	T
F	F	T	F	T

So, $((A \Rightarrow B) \Rightarrow A) \Leftrightarrow A$ is a tautology.

So, $(A \Rightarrow B) \Rightarrow A$ and A are logically equivalent

(b), (c), (d) → Exercises