(e)

p	q	r	$q \leftrightarrow r$	$p \vee (q \leftrightarrow r)$	$p \vee q$	$p \vee r$	$(p\vee q) \leftrightarrow (p \vee r)$	$((p\vee(q\leftrightarrow r)) \leftrightarrow ((p\vee q)\leftrightarrow(p\vee r)))$
T	T	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T	T
T	F	T	F	T	T	T	T	T
F	F	T	F	F	F	T	F	T
T	T	F	F	T	T	T	T	T
F	T	F	F	F	T	F	F	T
T	F	F	T	T	T	T	T	T
F	F	F	T	T	F	F	T	T

So, $((p\vee(q\leftrightarrow r)) \leftrightarrow ((p\vee q)\leftrightarrow(p\vee r)))$ is a tautology.

So, $(p\vee(q\leftrightarrow r))$ and $(p\vee q)\leftrightarrow(p\vee r)$ are logically equivalent.

(f), (g) → Exercises

10. Prove

(a) $(A \Rightarrow B)$ is logically equivalent to $((\neg A)\vee B)$

(b) $(A \Rightarrow B)$ is logically equivalent to $(\neg(A \wedge (\neg B)))$

Solution: (a)

A	B	$\neg A$	$(\neg A)\vee B$	$A \Rightarrow B$	$(A\Rightarrow B) \Leftrightarrow ((\neg A)\vee B)$
T	T	F	T	T	T
F	T	T	T	T	T
T	F	F	F	F	T
F	F	T	T	T	T

So, $(A\Rightarrow B) \Leftrightarrow ((\neg A)\vee B)$ is a tautology.

So, $(A\Rightarrow B)$ is logically equivalent to $((\neg A)\vee B)$

(b) → Exercise

11. Prove that $\mathcal{B}$ is logically equivalent to $\mathcal{C}$ if and only if $\mathcal{B}$ logically implies $\mathcal{C}$ and $\mathcal{C}$ logically implies $\mathcal{B}$.

Solution: Let $\mathcal{B}$ is logically equivalent to $\mathcal{C}$. If $\mathcal{B} \Rightarrow \mathcal{C}$ takes value F then $\mathcal{B}$ takes the value T and $\mathcal{C}$ takes the value T, a contradiction to the fact $\mathcal{B}$ and $\mathcal{C}$ both takes the same value as they are logically equivalent. So $\mathcal{B}$ logically implies $\mathcal{C}$. Similarly, $\mathcal{C}$ logically implies $\mathcal{B}$.

Conversely, let $\mathcal{B}$ logically implies $\mathcal{C}$ and $\mathcal{C}$ logically implies $\mathcal{B}$. If $\mathcal{B} \Leftrightarrow \mathcal{C}$ takes value F then either $\mathcal{B}$ takes the value T and $\mathcal{C}$ takes the value F or $\mathcal{B}$ takes the value F and $\mathcal{C}$ takes the value T. In the first case $\mathcal{B} \Rightarrow \mathcal{C}$ takes the value F and in second case $\mathcal{C} \Rightarrow \mathcal{B}$ takes the value F, both are contradiction to the fact that $\mathcal{B}$ logically implies $\mathcal{C}$ and $\mathcal{C}$ logically implies $\mathcal{B}$. So, $\mathcal{B} \Rightarrow \mathcal{C}$ is always true. So $\mathcal{B}$ and $\mathcal{C}$ are logically equivalent.

12. Show that $\mathcal{B}$ and $\mathcal{C}$ are logically equivalent if and only if, in their truth tables, the columns under $\mathcal{B}$ and $\mathcal{C}$ are the same.

Proof: $\mathcal{B}$ and $\mathcal{C}$ are logically equivalent if and only if $\mathcal{B} \Leftrightarrow \mathcal{C}$ is a tautology. So, $\mathcal{B}$ and $\mathcal{C}$ are logically equivalent if and only if $\mathcal{B} \Leftrightarrow \mathcal{C}$ always takes the value T. So, $\mathcal{B}$ and $\mathcal{C}$ are logically equivalent if and only if ~~both~~ either both $\mathcal{B}$ and $\mathcal{C}$ take the value F or both take the value T. So, $\mathcal{B}$ and $\mathcal{C}$ are logically equivalent if and only if, in their truth tables, the column under $\mathcal{B}$ and $\mathcal{C}$ are the same.

13. Prove that $\mathcal{B}$ and $\mathcal{C}$ are logically equivalent if and only if $(\neg\mathcal{B})$ and $(\neg\mathcal{C})$ are logically equivalent.

Ans: <u>Exercise</u>

13. Which of the following statements are logically implied by $(A \wedge B)$?

(a) A
(b) B
(c) $(A \vee B)$
(d) $((\neg A) \vee B)$
(e) $((\neg B) \Rightarrow A)$
(f) $(A \Leftrightarrow B)$
(g) $(A \Rightarrow B)$
(h) $((\neg B) \Rightarrow (\neg A))$
i. $(A \wedge (\neg B))$

Solution: (a)

A	B	$A \wedge B$	$(A \wedge B) \Rightarrow A$
T	T	T	T
F	T	F	T
T	F	F	T
F	F	F	T

So, $(A \wedge B) \Rightarrow A$ is a tautology. So, $(A \wedge B)$ logically implies A or. A is logically implied by $(A \wedge B)$

(b), (c) – Exercises

(d)

A	B	$\neg A$	$(\neg A) \vee B$	$A \wedge B$	$A \wedge B \Rightarrow ((\neg A) \vee B)$
T	T	F	T	T	T
F	T	T	T	F	T
T	F	F	F	F	T
F	F	T	T	F	T

So, $A \wedge B \Rightarrow ((\neg A) \vee B)$ is a tautology. So $(\neg A) \vee B$ is logically implied by $A \wedge B$

(e)

A	B	$\neg B$	$\neg B \Rightarrow A$	$A \wedge B$	$A \wedge B \Rightarrow (\neg B \Rightarrow A)$
T	T	F	T	T	T
F	T	F	T	F	T
T	F	T	T	F	T
F	F	T	F	F	T

So, $A \wedge B \Rightarrow (\neg B \Rightarrow A)$ is a tautology. So $(\neg B \Rightarrow A)$ is logically implied by $A \wedge B$.

(f), (g), (h), (i) — Exercises

14. Repeat exercise 13 with $(A \wedge B)$ replaced by $(A \Rightarrow B)$ and by $(\neg(A \Rightarrow B))$.

Ans: (a), (b) · Exercise

(c)

A	B	$A \vee B$	$A \Rightarrow B$	$(A \Rightarrow B) \Rightarrow (A \vee B)$
T	T	T	T	T
F	T	T	T	T
T	F	T	F	T
F	F	F	T	F

As there is one F in the column corresponding to $(A \Rightarrow B) \Rightarrow (A \vee B)$ is not a tautology.

So, $(A \Rightarrow B) \Rightarrow (A \vee B)$ is not a tautology.

So, $A \vee B$ is not logically implied by $A \Rightarrow B$

Again,

A	B	$A \vee B$	$A \Rightarrow B$	$\neg(A \Rightarrow B)$	$(\neg(A \Rightarrow B)) \Rightarrow (A \vee B)$
T	T	T	T	F	T
F	T	T	T	F	T
T	F	T	F	T	T
F	F	F	T	F	T

So, $(\neg(A \Rightarrow B)) \Rightarrow (A \vee B)$ is a tautology. So,

$A \vee B$ is logically implied by $\neg(A \Rightarrow B)$

(d), (e), (f), (g), (h), (i) → Exercises

15. Repeat exercise 13 with $(A \wedge B)$ replaced by $(A \vee B)$

Ans: Exercise

16. Repeat exercise 13 with $(A \wedge B)$ replaced by $(A \Leftrightarrow B)$ and by $(\neg(A \Leftrightarrow B))$

Ans. Exercise.

16. Determine whether each of the following is a tautology, is contradictory, or neither.

(a) $B \Leftrightarrow (B \vee B)$

(b) $((A \Rightarrow B) \wedge B) \Rightarrow A$

(c) $(\neg A) \Rightarrow (A \wedge B)$

(d) $(p \to q) \to ((q \to r) \to (p \to r))$

(e) $(A \Leftrightarrow \neg B) \Rightarrow A \vee B$

(f) $A \wedge (\neg(A \vee B))$

(g) $(p \to q) \leftrightarrow ((\sim p) \vee q)$

(h) $(A \Rightarrow B) \Leftrightarrow \neg(A \wedge (\neg B))$

(i) $(B \Leftrightarrow (B \Leftrightarrow A)) \Rightarrow A$

(j) $A \wedge \neg A \Rightarrow B$

Ans: (a), (b), (c) → Exercises

(d)

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(q \to r) \to (p \to r)$	$(p \to q) \to ((q \to r) \to (p \to r))$
T	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T
T	F	T	F	T	T	T	T
F	F	T	T	T	T	T	T
T	T	F	T	F	F	T	T
F	T	F	T	F	T	T	T
T	F	F	F	T	F	F	T
F	F	F	T	T	T	T	T

So, $(p \to q) \to ((q \to r) \to (p \to r))$ is a tautology.

(e)

A	B	$\neg B$	$A \Leftrightarrow \neg B$	$A \vee B$	$(A \Leftrightarrow \neg B) \Rightarrow A \vee B$
T	T	F	F	T	T
F	T	F	T	T	T
T	F	T	T	T	T
F	F	T	F	F	T

So, $(A \Leftrightarrow \neg B) \Rightarrow (A \vee B)$ is a tautology

(f), (g), (h), (i) → Exercises

(j)

A	B	$\neg A$	$A \wedge \neg A$	$A \wedge \neg A \Rightarrow B$
T	T	F	F	T
F	T	T	F	T
T	F	F	F	T
F	F	T	F	T

So, $A \wedge \neg A \Rightarrow B$ is a tautology