9. Check the validity of the following arguments: "If my program runs successfully then I will submit my project. I submit my project only if I can appear in the examination. Either my program runs successfully or the computer crashes. Therefore, if the computer does not crash then I can appear in the examination."

Solution: Let p: My program runs successfully

q: I submit my project

r: I appear in the examination

s: Computer crashes

Then the given argument is

$$\begin{array}{c} p \to q \\ q \to r \\ p \vee s \\ \hline \sim s \to r \end{array}$$

The steps for deduction that $\sim s \to r$ is a valid conclusion or not from the above premises are as follows:

1. $p \to q$ Rule P
2. $q \to r$ Rule P
3. $p \to r$ Rule T, 1, 2 and Hypothetical Syllogism
4. $p \vee s$ Rule P
5. $\sim p \to s$ Rule T, 4 and the equivalence $p \to q \equiv \sim p \vee q$ and $\sim(\sim p) \equiv p$
6. $\sim s \to p$ Rule T and the equivalence $p \to q \equiv \sim q \to \sim p$
7. $\sim s \to r$ Rule T, 3, 6 and Hypothetical Syllogism

Hence $\sim s \to r$ is a valid conclusion from the given premises. Hence, the given argument is valid.

10. Check the validity of the following arguments: "If I pass M.Sc. with high marks, I will be assured a good job. If I am assured a good job then ~~If~~ my father will be happy. My father is not happy. Therefore, I did not pass M.Sc. with high marks."

Solution: Let p: I pass M.Sc. with high marks

q: I am assured a good job

r: My father is happy

Then the given argument is

$$\begin{array}{c} p \to q \\ q \to r \\ \sim r \\ \hline \sim p \end{array}$$

The steps for deduction that $\sim p$ is a valid conclusion or not from the above premises are as follows:

1. $p \to q$ Rule P
2. $q \to r$ Rule P
3. $p \to r$ Rule T, 1, 2 and Hypothetical Syllogism
4. $\sim r$ Rule P
5. $\sim p$ Rule T, (3), (4) and Modus Tollens.

Hence $\sim p$ is a valid conclusion. It follows that the given argument is valid

Inconsistency of premises: A set of premises $p_1, p_2, \ldots, p_n$ is said to be inconsistent if their conjunction implies a contradiction.

Example 1 Show that the following set of premises is ~~inform~~ inconsistent: " If the contract is valid, then Bipin is liable for penalty. If Bipin is liable for penalty, he will go bankrupt. If the bank will loan him money, he will not go bankrupt. As a matter fact, the contract is valid and the bank will loan him money

Solution: Let p: The contract is valid

q: Bipin is liable for penalty

r: Bank will loan him money

s: He will go bankrupt.

Then the given premises are

$p \to q,\ q \to s,\ r \to \sim s,\ p \wedge r$

The steps for validating whether the given premises are inconsistent or not are as follows:

1. $p \to q$ Rule P
2. $q \to s$ Rule P
3. $p \to s$ Rule T, 1, 2 and Hypothetical Syllogism.
4. $p \wedge r$ Rule P
5. p Rule T and 4
6. r Rule T and 4
7. s Rule T, 3, 5, and Modus Ponens
8. $r \to \sim s$ Rule P
9. $\sim s$ Rule T, 6, 8 and Modus Ponens
10. $s \wedge \sim s$ Rule T, 7, 9

Hence the conclusion is a contradiction

This gives that the set of premises is ~~co~~ inconsistent.

Direct Proof

A direct proof is a proof in which the truth of the premises $p_1, p_2, \ldots, p_n$ directly shows the conclusion p, i.e., it shows that

$p_1 \wedge p_2 \wedge \cdots \wedge p_n \to p$ is a tautology.

Example Prove that the product of two odd integers is always odd.

Solution: Let p: m is an odd integer

q: n is an odd integer

r: mn is an odd integer

Now since m and n are odd integers $\exists$ integers k_1, k_2 such that $m = 2k_1+1$, $n = 2k_2+1$

$\therefore mn = (2k_1+1)(2k_2+1) = 2(2k_1k_2 + k_1 + k_2) + 1$

$= 2k_3 + 1$ where $k_3 = 2k_1k_2 + k_1 + k_2$ is an integer

So, mn is an odd integer. Hence $p \wedge q \to r$ is a tautology.

Indirect Proof To prove $p \to q$ indirectly, we assume $\sim q$ and show that $\sim p$. This uses the fact that $p \to q \equiv \sim q \to \sim p$

Example: Prove that n^2 is odd when n is also odd where n is an integer

Proof: Let p: n^2 is odd

q: n is odd

Then we have to prove $p \to q$ is true whenever p and q are both true

Here, we use indirect proof method

We shall show $\sim q \to \sim p$

Now, let $\sim q$ be true

So, n is not odd, i.e., n is even

Let $n = 2k$, k is an integer

Then $n^2 = 4k^2 = 2k'$ where $k' = 2k^2$ is an integer.

So, $\sim p$ is true

So, $\sim q \to \sim p$ is true

So, $p \to q$ is true.

Proof by Contradiction

A proof by contradiction means the argument

$$\begin{array}{l} p \to q \\ \sim q \\ \hline \sim p \end{array}$$

Example: Prove that $\sqrt{2}$ is not a rational number

Solution: Let q: $\sqrt{2}$ is not a rational number

Then $\sim q$: $\sqrt{2}$ is a rational number.

Let us suppose $\sim q$ is true. Then $\exists$ integers a and b such that $\sqrt{2} = \frac{a}{b}$ and $\gcd(a, b) = 1$

So, $a^2 = 2b^2$. So, a^2 is even. So, a is even. Let $a = 2k$, k is an integer. So, $4k^2 = 2b^2$ or, $b^2 = 2k^2$. So, b^2 is even. So, b is even. As a and b are both even then $\gcd(a,b) = 1$ is not ~~true~~ true. So our assumption is wrong. So, q is true. Hence $\sqrt{2}$ is not a rational number.

<u>Proof by cases</u> : A proof that utilises the fact that

$$p_1 \vee p_2 \vee \cdots \vee p_n \to p \equiv (p_1 \to p) \wedge (p_2 \to p) \wedge \cdots \wedge (p_n \to p)$$

where $p_1, p_2, \ldots, p_n$ are hypotheses and p is the conclusion.

<u>Example</u> Prove that for every positive integer n, $n^3 + n$ is always an even integer.

Solution: Let p_1 : n is a positive even integer

p_2 : n is a positive odd integer

p : $n^3 + n$ is an even integer.

We will show that $p_1 \vee p_2 \to p$ is true.

<u>Case 1</u> We shall show that $p_1 \to p$ is true

Let p_1 be true. Then n is a positive even integer

Let $n = 2k$, k is an integer.

Now $n^3 + n = 8k^3 + 2k = 2(4k^3 + k) = 2k_1$ where $k_1 = 4k^3 + k$ is an integer. So $n^3 + n$ is an even integer

Hence $p_1 \to p$ is true

<u>Case 2</u> We shall show that $p_2 \to p$ is true

Let p_2 be true. Then n is an odd positive integer

So, $n = 2k+1$, k is an integer

Then $n^3 + n = (2k+1)^3 + (2k+1)$

$= (8k^3 + 12k^2 + 6k + 1) + (2k+1)$

$= 2(4k^3 + 6k^2 + 4k + 1) = 2k_2$ where

$k_2 = 4k^3 + 6k^2 + 4k + 1$ is an integer

So, $n^3 + n$ is an even integer. So $p_2 \to p$ is true. So, from Case 1 & 2 $(p_1 \to p) \wedge (p_2 \to p)$ is true. Hence $p_1 \vee p_2 \to p$ is true as $(p_1 \vee p_2 \to p) \equiv (p_1 \to p) \wedge (p_2 \to p)$