Proof by Principles of Mathematical Induction :

1. First principle of induction :

Let S be a subset of $\mathbb{N}$ such that

(i) $1 \in S$

(ii) If $m \in S$ then $m+1 \in S$

Then $S = \mathbb{N}$ (the set of all natural numbers)

(Alternative form: Let $P(n)$ be a statement such that $P(1)$ is true and $P(m)$ is true $\Rightarrow P(m+1)$ is true then $P(n)$ is true for all $n \in \mathbb{N}$)

Example 1 Prove by mathematical induction $1^3+2^3+\dots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$, $\forall n \in \mathbb{N}$

Proof:

Let $P(n)$ be the statement $1^3+2^3+\dots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$

For $n=1$, L.H.S $= 1$ and R.H.S $= \left[\frac{1(1+1)}{2}\right]^2 = 1$

So $P(n)$ is true for $n=1$

Let $P(m)$ be true. So, $1^3+2^3+\dots+m^3 = \left[\frac{m(m+1)}{2}\right]^2 \quad \dots (i)$

Then $1^3+2^3+\dots+m^3+(m+1)^3 = \left(\frac{m(m+1)}{2}\right)^2 + (m+1)^3$ by (i)

$= (m+1)^2\left[\left(\frac{m}{2}\right)^2 + m+1\right]$

$= (m+1)^2\left[\frac{m^2+4m+4}{4}\right]$

$= (m+1)^2\,\frac{(m+2)^2}{4}$

$= \left(\frac{(m+1)(m+1+1)}{2}\right)^2$

So, $P(m+1)$ is true

So, $P(n)$ is true for all $n \in \mathbb{N}$

So, $1^3+2^3+\dots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$, $\forall n \in \mathbb{N}$

2. ~~Principle~~ & Second Principle of Mathematical Induction :

Let $S \subset \mathbb{N}$ be such that

(i) $1 \in S$

(ii) $1, 2, \dots, k \in S \Rightarrow k+1 \in S$

Then $S = \mathbb{N}$

(Alternative form, Let $P(n)$ be a statement such that $P(1)$ is true and $P(1), P(2), \dots, P(k)$ are true $\Rightarrow P(k+1)$ is true Then $P(n)$ is true for all $n \in \mathbb{N}$)

Example 2 Prove that $(3+\sqrt{7})^n + (3-\sqrt{7})^n$ is an even integer, for all $n \in \mathbb{N}$

Proof: Let $P(n)$ be the statement $(3+\sqrt{7})^n + (3-\sqrt{7})^n$ is an even integer

for $n=1$ $(3+\sqrt{7})^1 + (3-\sqrt{7})^1 = 6$, which is an even integer

So, $P(1)$ is true

Let $P(1), P(2), \ldots P(k)$ be true

now $(3+\sqrt{7})^{k+1} + (3-\sqrt{7})^{k+1} = \alpha^{k+1} + \beta^{k+1}$ where $\alpha = 3+\sqrt{7}$; $\beta = 3-\sqrt{7}$

$= (\alpha+\beta)(\alpha^k+\beta^k) - \alpha\beta(\alpha^{k-1}+\beta^{k-1})$

$= 6(\alpha^k+\beta^k) - 2(\alpha^{k-1}+\beta^{k-1})$ as $\alpha+\beta = 6$ and $\alpha\beta = 2$

As $\alpha^k+\beta^k$ and $\alpha^{k-1}+\beta^{k-1}$ are both even integers

So $(3+\sqrt{7})^{k+1} + (3-\sqrt{7})^{k+1}$ is even. So $P(k+1)$ is true

So by second principle of induction, $P(n)$ is true for all $n \in \mathbb{N}$.

So, $(3+\sqrt{7})^n + (3-\sqrt{7})^n$ is even for all $n \in \mathbb{N}$

Some Problems regarding dnf and cnf (dnf is also written as DNF & cnf is also written as CNF, full dnf is also written as Principal dnf and notation becomes PDNF. Similarly PCNF.)

Exercise 1 Find the DNF and CNF of $P(p,q,r)$ from the following truth table

p	q	r	$P(p,q,r)$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

[Note: Let p and q be any two propositions. Then $p\wedge q$, $p\wedge\sim q$, $\sim p\wedge q$ and $\sim p\wedge\sim q$ are called minterms of p and q.

If p, q, r are three propositions, then $p\wedge q\wedge r$, $\sim p\wedge q\wedge r$, $p\wedge\sim q\wedge r$, $\sim p\wedge\sim q\wedge r$, $p\wedge q\wedge\sim r$, $\sim p\wedge q\wedge\sim r$, $p\wedge\sim q\wedge\sim r$ and $\sim p\wedge\sim q\wedge\sim r$ are the minterms for p, q and r

Similarly, If p and q are statements then $p\vee q$, $p\vee\sim q$, $\sim p\vee q$ and $\sim p\vee\sim q$ are call maxterms.

If p, q, r are propositions (or statements) then $p\vee q\vee r$, $\sim p\vee q\vee r$, $p\vee\sim q\vee r$, $\sim p\vee\sim q\vee r$, $p\vee q\vee\sim r$, $\sim p\vee q\vee\sim r$, $p\vee\sim q\vee\sim r$ and $\sim p\vee\sim q\vee\sim r$ are the maxterms for p, q and r]

Solution: Here we see that there are four truth values of $P(p,q,r)$ which are T and so corresponding to them, the minterms are $p\wedge q\wedge r$, $p\wedge\sim q\wedge r$, $p\wedge\sim q\wedge\sim r$ and $\sim p\wedge\sim q\wedge r$.

Hence the DNF of $P(p,q,r)$ is

$(p\wedge q\wedge r)\vee(p\wedge\sim q\wedge r)\vee(p\wedge\sim q\wedge\sim r)\vee(\sim p\wedge\sim q\wedge r)$

i.e. $P(p,q,r)\equiv(p\wedge q\wedge r)\vee(p\wedge\sim q\wedge r)\vee(p\wedge\sim q\wedge\sim r)\vee(\sim p\wedge\sim q\wedge r)$

Again, we see that there are four truth values of $P(p,q,r)$ which are F and so, corresponding to them, the maxterms are

$\sim p\vee\sim q\vee r$, $p\vee\sim q\vee\sim r$, $p\vee\sim q\vee r$ and $p\vee q\vee r$

Hence the CNF of $P(p,q,r)$ is

$(\sim p\vee\sim q\vee r)\wedge(p\vee\sim q\vee\sim r)\wedge(p\vee\sim q\vee r)\wedge(p\vee q\vee r)$

i.e., $P(p,q,r)\equiv(\sim p\vee\sim q\vee r)\wedge(p\vee$

Exercise 2 Find the PDNF and PCNF of the statement form $Q(p,q,r)=p\wedge((q\leftrightarrow r)\vee(r\Leftrightarrow p))$

Solution: Let us construct the truth table for $Q(p,q,r)$ as follows:

p	q	r	$q \leftrightarrow r$	$r \leftrightarrow p$	$(q \leftrightarrow r) \vee (r \leftrightarrow p)$	$Q(p,q,r)$
T	T	T	T	T	T	T
T	T	F	F	F	F	F
T	F	T	F	T	T	T
T	F	F	T	F	T	T
F	T	T	T	F	T	F
F	T	F	F	T	T	F
F	F	T	F	F	F	F
F	F	F	T	T	T	F

By choosing the minterms corresponding to truth values of $Q(p,q,r)$ which are T, we have,

$Q(p,q,r) \equiv (p \wedge q \wedge r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r)$ which is the required PDNF

Again, by choosing the maxterms corresponding to truth values of $Q(p,q,r)$ which are F, we have

$$Q(p,q,r) \equiv (\sim p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge (p \vee \sim q \vee r) \wedge (p \vee q \vee \sim r) \wedge (p \vee q \vee r)$$ which is the required PCNF

Exercise 3 Without using truth, find the PDNF of the following statement forms:

(i) $((\sim p \wedge q) \vee \sim (p \wedge r)) \wedge \sim (p \vee (q \wedge r))$

(ii) $\sim ((p \wedge q) \vee (p \wedge \sim r)) \vee \sim q$

Solution (i) $((\sim p \wedge q) \vee \sim (p \wedge r)) \wedge \sim (p \vee (q \wedge r))$

$\equiv ((\sim p \wedge q) \vee \sim p \vee \sim r) \wedge \sim p \wedge \sim (q \wedge r)$ [De Morgan's Law]

$\equiv ((\sim p \wedge q) \vee \sim p \vee \sim r) \wedge \sim p \wedge (\sim q \vee \sim r)$ [De Morgan's Law]

$\equiv (((\sim p \wedge q) \vee \sim p) \vee \sim r) \wedge \sim p \wedge (\sim q \vee \sim r))$ (Associative Law)

$\equiv (\sim p \vee \sim r) \wedge \sim p \wedge (\sim q \vee \sim r)$ (Absorption Law)

$\equiv ((\sim p \vee \sim r) \wedge \sim p) \wedge (\sim q \vee \sim r)$ (Associative law)

$\equiv \sim p \wedge (\sim q \vee \sim r)$ (Absorption Law)

$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)$ (Distributive law)

$\equiv ((\sim p \wedge \sim q) \wedge (r \vee \sim r)) \vee ((\sim p \wedge \sim r) \wedge (q \vee \sim q))$ $[\because a \vee \sim a \equiv T$ and $a \wedge T \equiv a]$

$\equiv (\sim p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge \sim r \wedge q) \vee (\sim p \wedge \sim r \wedge \sim q)$

which is the required PDNF

(ii) $\sim((p \wedge q) \vee (p \wedge \sim r)) \vee \sim q$

$\equiv (\sim(p \wedge q) \wedge \sim(p \wedge \sim r)) \vee \sim q$ (De Morgan's Law)

$\equiv ((\sim p \vee \sim q) \wedge (\sim p \vee r)) \vee \sim q$ (De Morgan's law)

$\equiv \sim p \vee (\sim q \wedge r) \vee \sim q$ (Distributive Law)

$\equiv (\sim p \wedge (q \vee \sim q)) \vee (\sim q \wedge r \wedge (p \vee \sim p)) \vee (\sim q \wedge (p \vee \sim p))$

$[\because a \vee \sim a \equiv T$ and $a \wedge T \equiv a]$

$\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (\sim q \wedge r \wedge p) \vee (\sim q \wedge r \wedge \sim p)$
$\vee (\sim q \wedge p) \vee (\sim q \wedge \sim p)$
$\vee (\sim q \wedge r \wedge p)$ (Distributive Laws)

$\equiv (\sim p \wedge q \wedge (r \vee \sim r)) \vee (\sim p \wedge \sim q \wedge (r \vee \sim r)) \vee (\sim q \wedge r \wedge \sim p) \vee (\sim q \wedge p \wedge (r \vee \sim r))$
$\vee (\sim q \wedge \sim p \wedge (r \vee \sim r))$

$[\because a \vee \sim a \equiv T$ and $a \wedge T \equiv a]$

$\equiv (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge \sim r) \vee (\sim q \wedge r \wedge \sim p) \vee$
$(\sim q \wedge r \wedge p) \vee (\sim q \wedge p \wedge r) \vee (\sim q \wedge p \wedge \sim r) \vee (\sim q \wedge \sim p \wedge r) \vee (\sim q \wedge \sim p \wedge \sim r)$

[Distributive Laws]

$\equiv (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge \sim r) \vee (\sim q \wedge r \wedge p) \vee (\sim q \vee p \wedge \sim r)$

(Absorption Laws)

Which is the required PDNF