SEMESTER-IV

LECTURE NOTES ON

PARTIAL DIFFERENTIAL EQUATIONS

1ST PART

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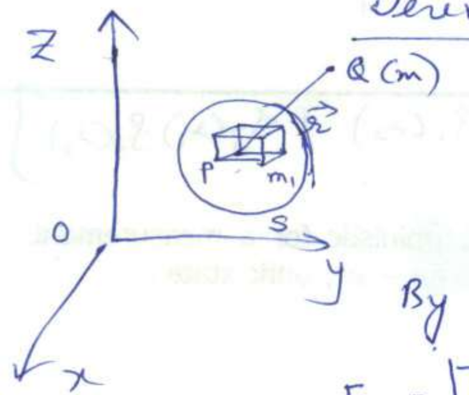
REFERENCE BOOK: **PARTIAL DIFFERENTIAL EQUATIONS BY SANKARA RAO**

Derivation of Laplace Eqn (1)



Let $\exists$ 2 particles m at Q and m_1 at P

$$\vec{PQ} = \vec{r}.$$

By Newton's gravitational law,

$$F = |\vec{F}| = G\frac{m_1 m}{r^2} \quad \text{where } r = |\vec{r}|$$

$G \to$ gravitational constant.

If $m = 1$ ~~and~~ $\vec{PQ} = \vec{r}$ and $G = 1$ then the force at Q due to mass m_1 at P is given by $\vec{F} = -\frac{m_1}{r^2}\frac{\vec{r}}{r} = \frac{m_1}{r^3}\vec{r} = \vec{\nabla}\left(\frac{m_1}{r}\right)$ (−ve sign as force is attractive in nature) — (1)

which is called intensity of the gravitational force.

Now, Let a particle of unit mass moves under the attraction of a particle of mass m_1 at P from ∞ to Q.

$\therefore$ work done by $\vec{F}$ is $= \int_\infty^r \vec{F}\cdot d\vec{r} = \int_\infty^r \vec{\nabla}\left(\frac{m_1}{r}\right)\cdot d\vec{r}$

$$= m_1\int_\infty^r \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(r^{-1}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= m_1\int_\infty^r \left(\frac{\partial}{\partial x}(r^{-1})dx + \frac{\partial}{\partial y}(r^{-1})dy + \frac{\partial}{\partial z}(r^{-1})dz\right)$$

$$W = m_1\int_\infty^r d(r^{-1}) = \frac{m_1}{r} \quad \text{— (2)}$$

This work done (W) is defined to be the potential V at Q due to particle at P and conventionally denoted as,

$$V = -W = -\frac{m_1}{r} \quad \text{— (3)}$$

$\therefore$ eqn (1) becomes $\vec{F} = -\vec{\nabla}V$.

Case-I.

Now considering a system of particles of masses m_i at dist r_i ($i = 1, 2, \ldots, n$) from Q, then the force of attraction / mass at Q is ?

(2)

$$\vec{F} = \sum_{i=1}^{n} \nabla\left(\frac{m_i}{r_i}\right) = \nabla\left(\sum_{i=1}^{n} \frac{m_i}{r_i}\right)$$

∴ work done by the force acting on the particle is :-

$$V = -\int_{\infty}^{r} \vec{F}\cdot d\vec{r} = \sum_{i=1}^{n} \frac{m_i}{r_i} \quad (\text{using eq}^n \ (2))$$

$$\therefore \nabla^2 V = \vec{\nabla}\cdot(\vec{\nabla} V) = \vec{\nabla}\cdot\vec{\nabla}\left(\sum_{i=1}^{n} \frac{m_i}{r_i}\right) = \sum_{i=1}^{n} \nabla^2\left(\frac{m_i}{r_i}\right) = 0 \ (\because r_i \neq 0) \quad \text{--- (4)}$$

where $\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

Case-II Continuous distribution of matter of density ρ in a volume τ.

Here, $V(x,y,z)$ = Potential at a point $Q(\xi, \eta, \zeta)$ due to elementary mass at any pt $P(x,y,z)$ inside the volume τ is given by

$$V(x,y,z) = \iiint_{\tau} \frac{\rho(\xi,\eta,\zeta)}{r}\, d\tau \quad \text{where } r = |\vec{x}| = \left[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\right]^{1/2}$$

$$\therefore \nabla^2 V = \vec{\nabla}\cdot(\vec{\nabla} V) = \iiint_{\tau} \nabla^2\left(\frac{1}{r}\right) \rho(\xi,\eta,\zeta)\, d\tau \ (\because (\xi,\eta,\zeta) \text{ fixed point})$$

$$= 0 \quad \left(\because \nabla^2\left(\frac{1}{\{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\}^{1/2}}\right) = \nabla_1^2\left(\frac{1}{(x'^2+y'^2+z'^2)^{1/2}}\right)\right.$$

(where $\vec{\nabla}_1 = \hat{i}\frac{\partial}{\partial x'} + \hat{j}\frac{\partial}{\partial y'} + \hat{k}\frac{\partial}{\partial z'}$)

$$= \nabla_1^2\left(\frac{1}{r'}\right) = 0.$$

$\therefore \nabla^2 V = 0$ [Laplace Eqn]

Now proving $\nabla^2\left(\frac{1}{r}\right) = 0$.

$$\nabla^2\left(\frac{1}{r}\right) = \vec{\nabla}\cdot\left(\vec{\nabla}\left(\frac{1}{r}\right)\right) = \frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right)$$

Consider $\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = \frac{\partial^2}{\partial x^2}\left((x^2+y^2+z^2)^{-1/2}\right)$

$$= \frac{\partial}{\partial x}\left(-\frac{1}{2}\times\frac{2x}{(x^2+y^2+z^2)^{3/2}}\right) = -\frac{\partial}{\partial x}\left(\frac{x}{(\quad)^{3/2}}\right)$$

$$= -\frac{1}{(x^2+y^2+z^2)^{3/2}} + \frac{3\times 2x\cdot x}{2(x^2+y^2+z^2)^{5/2}} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$$

Similarly, $\frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) = \frac{2y^2-x^2-z^2}{(\quad)^{5/2}}$.

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = \frac{2z^2-x^2-y^2}{(\quad)^{5/2}}$$

$$\therefore \nabla^2\left(\frac{1}{r}\right) = \frac{2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2}{(\quad)^{5/2}} = 0.$$

4.1 OCCURRENCE OF THE WAVE EQUATION

One of the most important and typical homogeneous hyperbolic differential equ[...] wave equation. It is of the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

where c is the wave speed. This differential equation is used in many branches of Ph[...] Engineering and is seen in many situations such as transverse vibrations of [...] membrane, longitudinal vibrations in a bar, propagation of sound waves, electr[...] waves, sea waves, elastic waves in solids, and surface waves as in earthquakes. Th[...] of a wave equation is called a *wave function*.

An example for inhomogeneous wave equation is

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = F$$

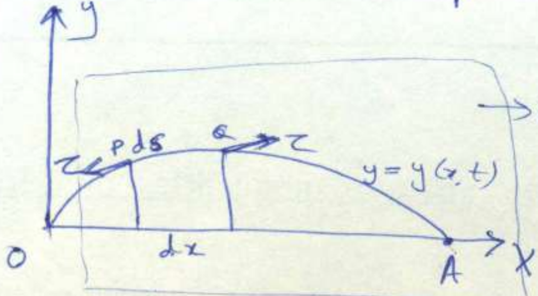
where F is a given function of spatial variables and time. In physical problems F [...] an external driving force such as gravity force. Another related equation is

$$\frac{\partial^2 u}{\partial t^2} + 2\gamma \frac{\partial u}{\partial t} - c^2 \nabla^2 u = F$$

where γ is a real positive constant. This equation is called a wave equation wi[...] term, the amplitude of which decreases exponentially as t increases. In Section 4[...] derive the partial differential equation describing the transverse vibration of a s[...]

(4) (3 → Pg 187 §4.1)

Derivation of one-dimensional wave eqn



(Drn of τ → ambiguous??)

→ motion takes place in this plane

↕ → dirn of particle's motion

Let a flexible string is stretched under tension τ b/w 2 pts O & A.

Assumptions: ① Motion takes place in one plane only & in this plane each particle moves in a dirn ⊥r to the eqm position of the string.

② τ is constant. ③ Gravitational force is neglected as compared with τ ④ Slope of deflection curve is small.

Let O, A → 2 fixed pts of the string $A \equiv (L, 0)$. O, A lies along the x-axis in its eqm position.

[theory used :- δ(Hamiltonian) = 0 ⇒ Hamiltonian → Stationary. For Hamiltonian, reqd T−V i.e. T, V. So T, V to be calculated & for V → elongation of the string reqd]

Let PQ be an infinitesimal segment PQ of the string.

Let $\rho \to$ mass/length of the string.

If the string is set vibrating in the xy plane, y → subsequent displacement of any pt P of the string from the eqm position of the string (i.e. X-axis)

Clearly $y \equiv y(x, t)$.

Now calculating elementary elongation dL

Under τ, dx stretched to ds. Now $ds = \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} \approx \left[1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right]dx$

(Taylor's expansion)

∴ elementary elongation $dL = ds - dx$

$$= \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx$$

Pot Energy / Work Done Calculation

∴ work done by this element dL against $\tau = \frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2 dx$

∴ total work done W (for whole string) is $= \frac{1}{2}\int_0^L \tau \left(\frac{\partial y}{\partial x}\right)^2 dx$ — ①

∴ V → Potential energy of the string $= W = \frac{1}{2}\int_0^L \tau\left(\frac{\partial y}{\partial x}\right)^2 dx$

Now,

Kinetic energy of the string (T) $= \frac{1}{2}\int_0^L \tau\left(\frac{\partial y}{\partial t}\right)^2 dx$ — ②

$\left(\frac{\partial y}{\partial t}\right)$ ↳ (velocity)^2.

Using ~~Hamilt~~ Hamilton's Principle $\delta \left(\int_{t_0}^{t_1} (T-V)\,dt \right) = 0$ (5)

$\Rightarrow \int_{t_0}^{t_1} (T-V)\,dt$ is stationary.

$\therefore \frac{1}{2}\int_{t_0}^{t_1}\left[\rho\left(\frac{\partial y}{\partial t}\right)^2 - \tau\left(\frac{\partial y}{\partial x}\right)^2\right]dx\,dt$ is stationary. & is of the form

$\iint F(x, y, t, y_x, y_t)\,dx\,dt$. ($x, t \to$ indep variables)

By Euler-Ostrogradsky eq^n,

$$\frac{\partial F}{\partial y} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial y_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y_x}\right) = 0 \quad - (3)$$

where $F = \frac{1}{2}\;\rho\left(\frac{\partial y}{\partial t}\right)^2 - \tau\left(\frac{\partial y}{\partial x}\right)^2$

$$\therefore \frac{\partial F}{\partial y_t} = \frac{2\rho}{2}\left(\frac{\partial y}{\partial t}\right) - 0 = \rho\left(\frac{\partial y}{\partial t}\right)$$

$$\frac{\partial F}{\partial y_x} = \tau\left(\frac{\partial y}{\partial x}\right) \text{ and } \frac{\partial F}{\partial y} = 0.$$

$\therefore$ eq^n (3) becomes

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial y}{\partial t}\right) - \frac{\partial}{\partial x}\left(\tau\frac{\partial y}{\partial x}\right) = 0 \quad - (4)$$

If the string is homogeneous, then ρ & τ are constants

$\therefore$ eq^n (4) becomes

$$\rho\frac{\partial^2 y}{\partial t^2} = \tau\frac{\partial^2 y}{\partial x^2} \Rightarrow \boxed{\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}} \quad - (5)$$

where $\boxed{c^2 = \tau/\rho}$

4.3 SOLUTION OF ONE-DIMENSIONAL WAVE EQUATION BY CANONICAL REDUCTION

The one-dimensional wave equation is

$$u_{tt} - c^2 u_{xx} = 0 \tag{4.6}$$

Choosing the characteristic lines

$$\xi = x - ct, \quad \eta = x + ct \tag{4.7}$$

the chain rule of partial differentiation gives

$$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$$

$$u_t = u_\xi \xi_t + u_\eta \eta_t = c(u_\eta - u_\xi)$$

In the operator notation we have

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta},\quad \frac{\partial}{\partial t}=c\left(\frac{\partial}{\partial \eta}-\frac{\partial}{\partial \xi}\right)$$

Thus, we get

$$\frac{\partial^2 u}{\partial x^2}=\left(\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}\right)^2 u=u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta} \tag{4.8}$$

$$\frac{\partial^2 u}{\partial t^2}=c^2(u_{\xi\xi}-2u_{\xi\eta}+u_{\eta\eta}) \tag{4.9}$$

Substituting Eqs. (4.8) and (4.9) into Eq. (4.6), we obtain

$$4u_{\xi\eta}=0 \tag{4.10}$$

Integrating, we get

$$u(\xi,\eta)=\phi(\xi)+\psi(\eta),$$

where ϕ and ψ are arbitrary functions. Replacing ξ and η as defined in Eq. (4.7), we have the general solution of the wave equation (4.6) in the form

$$u(x,t)=\phi(x-ct)+\psi(x+ct) \tag{4.11}$$

Interpreting sol^n $\quad u(x,t) = \underbrace{\phi(x-ct)}_{u_1} + \underbrace{\psi(x+ct)}_{u_2}$

$u_1 = \phi(x-ct)$ (Interpreting sol^n)

↳ Represents a travelling wave which is moving to the right (+ve dir^n of X-axis) with speed c. Its shape given by function ϕ. $\therefore$ clearly shape remains unchanged (as $u_1 = \phi(x-ct)$) for all time t. Only the wave profile (given by ϕ) shifts to the right in course of time.

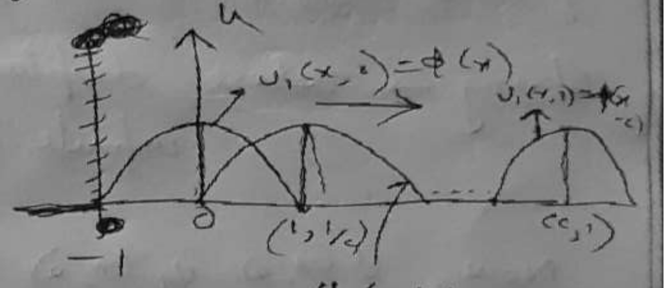
Put $t = 0, 1/c, 1, \ldots$

$u_1(x,0) = \phi(x)$

$u_1(x, 1/c) = \phi(x-1)$

$u_1(x, 1) = \phi(x-c)$

$\vdots$

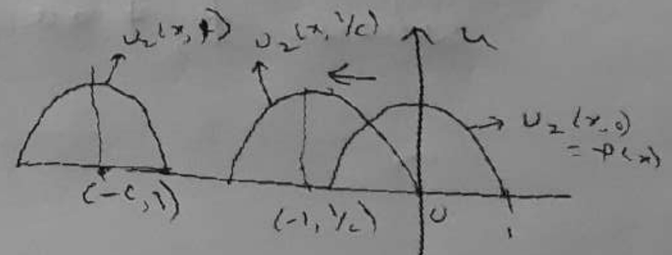


Travelling Wave Profile.

Similarly

$u_2 = \psi(x+ct)$

Represents wave travelling along $-ve$ dir^n of X-axis (towards left)



Initial Value Problem: D' Alembert's Sol^n

Cauchy type $\rightarrow$ $\begin{cases} u_{tt} - c^2 u_{xx} = 0 \quad -\infty < x < \infty, \ t \geq 0 & \text{— (1)} \\ \text{Initial condition} \quad u(x,0) = \eta(x) \ ; \ u_t(x,0) = \nu(x) & \text{— (2)} \end{cases}$

Curves $u(x,0)$, $u_t(x,0)$ on which the initial data $\eta(x)$ and $\nu(x)$ are prescribed are X-axis

$\eta(x)$ & $\nu(x) \Rightarrow$ 2 times continuously diff'ble.

String $\rightarrow$ whose displacement given by (x,t) has infinite length.

Now, $u(x,0)$ $u_t(x,0)$ give initial displacement & velocity

General sol^n : $\rightarrow u(x,t) = \phi(x+ct) + \psi(x-ct)$ — (3)

where ϕ, ψ are arbitrary functions.

$\therefore\ u_t(x,0) = c[\phi'(x) - \psi'(x)] = \nu(x)$ (by ②) — (3)

Also $\phi(x) + \psi(x) = \eta(x)$ (by ②) — (4)

Integrating eqn (3) we have

$$\phi(x) - \psi(x) = \frac{1}{c}\int_0^x \nu(\xi)\,d\xi \quad — (5)$$

Adding & subtracting eqⁿ (5) & eqⁿ (4)

$$\phi(x) = \frac{\eta(x)}{2} + \frac{1}{2c}\int_0^x \nu(\xi)\,d\xi \quad \text{—— (6)}$$

$$\psi(x) = \frac{\eta(x)}{2} - \frac{1}{2c}\int_0^x \nu(\xi)\,d\xi. \quad \text{— (7)}$$

Using (6) & (7) in eqⁿ (3) we get; reqd solⁿ:

$$\boxed{u(x,t) = \frac{1}{2}\left[\eta(x+ct) + \eta(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \nu(\xi)\,d\xi.}$$

D' Alembert's solution of One-Dimensional Wave Eqⁿ.

Clearly, if string is released from rest, i.e. $u_t(x,0) = 0$

i.e., $\nu(x) = 0$