Worked out Exercises 1.1.17 : 1. Let $\mathbb{R}$ be the set of all real numbers and R be the set of all real valued continuous functions defined on $\mathbb{R}$. Define $(f+g)(x) = f(x)+g(x)$ $(f.g)(x) = f(x)g(x)$ for all $f, g \in R$ and for all $x \in \mathbb{R}$. Show that $(R, +, .)$ is ~~a~~ a ring under the binary operations defined above.

Solution : Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2$, $x \in \mathbb{R}$. Then f is a continuous function. So, $f \in R$. So, R is non-empty. Again since the sum and product of two real valued continuous functions are again real valued continuous functions, we have, R is closed under $+$ and $.$. Now for any $x \in \mathbb{R}$ and for any $f, g, h \in R$, we have the following :

(i) $(f+g)(x) = f(x)+g(x) = g(x)+f(x) = (g+f)(x)$ which implies that $f+g = g+f$.

(ii) $((f+g)+h)(x) = (f+g)(x) + h(x) = \{f(x)+g(x)\} + h(x) = f(x) + (g(x)+h(x))$
$= \cancel{(f+(g+h))(x)} = f(x) + (g+h)(x) = (f+(g+h))(x)$. So,
$f+(g+h) = (f+g)+h$

(iii) The constant function 0 (which is defined as $0(x) = 0 \ \forall x \in \mathbb{R}$) ~~which~~ is in R and $(f+0)(x) = f(x) + 0(x) = f(x) + 0 = f(x)$ ~~for all $x \in \mathbb{R}$~~ for all $f \in R$. So, $f+0 = f$ for all $f \in R$.

(iv) For each $f \in R$, define a function $-f : \mathbb{R} \to \mathbb{R}$ by $(-f)(x) = -f(x)$ Then $-f \in R$ and
$(f+(-f))(x) = f(x) + (-f)(x) = f(x) - f(x) = 0 = 0(x)$
So, $f + (-f) = 0$

(v) $((f.g).h)(x) = (f.g)(x)\,h(x) = f(x)\,g(x)\,h(x) = f(x)\,(g(x)\,h(x))$
$= f(x)\,(g.h)(x) = (f.(g.h))(x)$. So, $(f.g).h = f.(g.h)$

(vi) $\left(f\cdot(g+h)\right)(x) = f(x)(g+h)(x)$

$= f(x)\left(g(x)+h(x)\right) = f(x)g(x) + f(x)h(x) = (f\cdot g)(x) + (f\cdot h)(x)$

$= \left((f\cdot g) + (f\cdot h)\right)(x)$ Hence $f\cdot(g+h) = f\cdot g + f\cdot h$

Similarly, we can show that $(g+h)\cdot f = g\cdot f + h\cdot f$

So $(R, +, \cdot)$ is a ring.

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2. Which of the following algebraic structures $(R, +, \cdot)$ forms a ring?

(i) Let X be any set and $R = \mathcal{P}(X)$, the power set of X. Define $A+B = A\Delta B$ and $A\cdot B = A\cap B$ for all $A, B\in R$

(ii) In the above set R, define $A+B = A\cup B$ and $A\cdot B = A\cap B$ for all $A, B\in R$

(iii) Let R be the set of all real valued continuous functions defined on $\mathbb{R}$. Define $(f+g)(x) = f(x)+g(x)$ $(f\circ g)(x) = f(g(x))$ for all $f, g\in R$ and for all $x\in\mathbb{R}$.

Solution: (i) First note that $R \neq \phi$ as $X\in R$. Let $A, B\in R = \mathcal{P}(X)$. Then $A\Delta B, A\cap B \in \mathcal{P}(X) = R$ which implies that R is closed under addition and multiplication. From set theory, we get the following results:

$$A\Delta B = B\Delta A \quad A\Delta(B\Delta C) = (A\Delta B)\Delta C, \quad A\Delta\phi = A$$

$$A\Delta A = \phi \quad A\cap(B\cap C) = (A\cap B)\cap C, \quad A\cap B = B\cap A$$

$$A\cap(B\Delta C) = (A\cap B)\Delta(A\cap C) \quad \text{for all } A, B, C\in R$$

So, R is a commutative ring. Also since $A\cap X = A$ for all $A\in R$, it follows that R has an ~~identity~~ unity, namely X.

Finally, Note that $A\cap A = A$ for all $A\in R$ shows that R is a Boolean Ring.

(ii) Note that the empty set ϕ is the identity with respect to addition as $A \cup \phi = A$ for all $A \in R$. But for any $\phi \neq A \in R$, there is no element $B \in R$ such that $A \cup B = \phi$.
Thus in this case $(R, +)$ is not a group unless $X = \phi$.
So, R is not a ring if $X \neq \phi$. Otherwise R is a trivial ring consisting of a single element namely ϕ.

(iii) It can be verified that R satisfies all the properties of a ring except distributive laws. For example, consider $f, g, h \in R$ defined by $f(x) = x^2$, $g(x) = 2x$, $h(x) = 3x$ Then $(f\circ(g+h))(x) = f(g(x) + h(x)) = f(2x+3x) = f(5x)$
$= 25x^2$ whereas, $(f\circ g + f\circ h)(x) = (f\circ g)(x) + (f\circ h)(x)$
$= f(g(x)) + f(h(x)) = f(2x) + f(3x) = 4x^2 + 9x^2 = 13x^2$.

So, $f\circ(g+h) \neq (f\circ g + f\circ h)$. So, R is not a ring.

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3. If $R = \{a, b, c, d\}$ be a ring, then complete the multiplication table of R, where

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

·	a	b	c	d
a	a	a	a	a
b	a	b		
c	a			a
d	a	b	c	

Is R commutative. Does it have a multiplicative identity.

Solution: First note that a is the additive identity and is the zero of the ring R. Also since $2b = 2c = 2d = a = 0$, we have $x = -x$ for all $x \in R$. Now $(b+c)b = b^2 + cb$ by distributive law. So, $db = b + cb$ or $b = b + cb$. This implies that $cb = 0 = a$
Next, consider the equality $d(c+d) = dc + d^2$. This implies that
$d^2 = d(c+d) - dc = db - c = b - c = b + c = d$

Similarly, $c(c+d) = c^2 + cd$ implies that $c^2 = 0 = a$ as $c(c+d) = cb = 0 = a = cd$

Again from $(b+c)c = bc + c^2$, it follows that

$c = dc = (b+c)c = bc + 0 = bc$. Finally, $bd = b(b+c) = b^2 + bc = b + c = d$

So, the completed multiplication table is

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	a	a	a
d	a	b	c	d

Clearly, R is not commutative as $bc = c \neq cb = a$.

Also it does not have an identity.

4. If R be a ring with unity such that $(xy)^2 = x^2y^2$ for all $x, y \in R$, then show that R is commutative

Solution: Let R be a ring with unity 1 such that $(xy)^2 = x^2y^2$ for all $x, y \in R$. Let $x, y \in R$. Now

$$(x(1+y))^2 = x^2(1+y)^2 \text{ which implies that}$$

$x^2 + xyx + x^2y + (xy)^2 = x^2 + 2x^2y + x^2y^2$. Since $(xy)^2 = x^2y^2$,

we get $x^2y - xyx = 0$. So, $x(xy - yx) = 0$ for all $x, y \in R$.

So, $(1+x)((1+x)y - y(1+x)) = 0$. This implies that $(1+x)(xy - yx) = 0$

or, $xy - yx + x(xy - yx) = 0$. So, $xy - yx = 0$ or $xy = yx$

for all $x, y \in R$. So, R is a commutative Ring.

Definition 1.1.18: Divisors of Zero: In a ring R, a non-zero element a is said to be a divisor of zero if there exists a non-zero element b in R such that $a.b = 0$, or a non-zero element c in R such that $c.a = 0$. In the first case a is said to be a left divisor of zero and in the second, a is said to be a right divisor of zero.