If, however R is a commutative ring, every left divisor of zero is also a right divisor of zero and conversely. So, there is no distinction between left and right divisors of zero in a commutative ring.

Examples 1.1.19 : 1. The ring $(\mathbb{Z},+,\cdot)$ contains no divisor of zero.

2. The rings $(\mathbb{Q},+,\cdot)$, $(\mathbb{R},+,\cdot)$ contain no divisor of zero.

3. In the ring $(\mathbb{Z}_6,+,\cdot)$, $\bar{2},\bar{3},\bar{4}$ are divisors of zero.

4. The ring $(\mathbb{Z}_5,+,\cdot)$ contains no divisor of zero.

5. In the ring $M_2(\mathbb{R})$, $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 2 & 6 \\ -1 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

So, $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ is a left divisor of zero and $\begin{pmatrix} 2 & 6 \\ -1 & -3 \end{pmatrix}$ is a right divisor of zero. Here $\begin{pmatrix} 2 & 6 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Again $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

So, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is both a left and a right divisor of zero. Similarly, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is also both a left and a right divisors of zero.

Definition 1.1.20 A ring R is said to satisfy

(i) the left cancellation law if $a \neq 0$, and $ab = ac \Rightarrow b = c$ for $a,b,c \in R$.

(ii) the right cancellation law if $a \neq 0$ and $ba = ca \Rightarrow b = c$ for $a,b,c \in R$

If both the cancellation laws hold, we say cancellation law holds.

In a ring cancellation law does not hold, in general.

For example, in the ring $(\mathbb{Z}_6,+,\cdot)$ $\bar{3}.\bar{2} = \bar{3}.\bar{4}$ but $\bar{2} \neq \bar{4}$

This shows that left cancellation law does not hold in $(\mathbb{Z}_6,+,\cdot)$

Also $\bar{2}\cdot\bar{3} = \bar{4}.\bar{3}$ but $\bar{2} \neq \bar{4}$. So, right cancellation law also does not hold in $(\mathbb{Z}_6,+,\cdot)$.

Theorem 1.1.21 The cancellation law holds in a ring R if and only if R has no divisor of zero.

Proof: Let R be a ring in which cancellation law holds.

Let $a, b \in R$ and $a \cdot b = 0$ where $a \neq 0$. Then $a \cdot b = 0 = a \cdot 0$

As the cancellation law holds, $b = 0$. So, a is not a left divisor of zero.

Let $a, b \in R$ and $a \cdot b = 0$ and $b \neq 0$. Then $a \cdot b = 0 = 0 \cdot b$

Since, the cancellation law holds in R, $a = 0$. This proves that b is not a right divisor of zero. So R has neither a left nor a right divisor of zero.

Conversely, let R be a ring containing no divisor of zero. Let $a \neq 0$ and $a \cdot b = a \cdot c$. Then $a \cdot (b - c) = 0$

Since R contains no divisor of zero, $b - c = 0$ or $b = c$

So, left cancellation law holds in R.

By similar argument, it can be proved that right cancellation law holds in R.

Definition 1.1.22 A non-trivial commutative ring R with unity is said to be an integral domain if it contains no divisor of zero.

Examples 1.1.23: 1. The rings $(\mathbb{Z}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$ are integral domains.

2. The ring $(2\mathbb{Z}, +, \cdot)$ is a commutative ring not containing divisors of zero. It is a ring without unity. So it is not an integral domain.

3. The ring $(\mathbb{Z}_5, +, \cdot)$ is a commutative ring with unity and the ring contains no divisor of zero. So, it is an integral domain.

4. The ring $(\mathbb{Z}_6, +, \cdot)$ is a commutative ring with unity. It contains divisor of zero. So, it is not an integral domain

5. The ring $M_n(\mathbb{Z})$ is a non-commutative ring. It is not an integral domain.

6. The ring $\mathbb{Z}\times\mathbb{Z}$ is a commutative ring with unity. $(1,0)$ and $(0,1)$ are non-zero elements of the ring and $(1,0)\cdot(0,1) = (0,0)$, the zero element of the ring. This shows that it contains divisors of zero. So, it is not an integral domain.

7. The set of all real valued continuous functions defined on the closed interval $[0,1]$ forms a ring. It is a commutative ring with unity, the unity being the constant function 1. The ring contains divisors of zero. For, let $f : [0,1] \to R$ be defined by
$$f(x) = 0 \quad x \in [0,\tfrac{1}{3})$$
$$= x - \tfrac{1}{3} \quad x \in [\tfrac{1}{3}, 1]$$
and $g : [0,1] \to R$ be defined by
$$g(x) = x - \tfrac{1}{3}, \ x \in [0, \tfrac{1}{3})$$
$$= 0 \quad x \in [\tfrac{1}{3}, 1]$$

Then f, g are real valued continuous functions on $[0,1]$ and $f \neq 0$ and $g \neq 0$ where 0 is the zero function then $f \cdot g = 0$.

Theorem 1.1.24 For any positive integer n, the ring $\mathbb{Z}_n$ of all integers modulo n is an integral domain if and only if n is a prime integer

Proof: Let $(\mathbb{Z}_n, +, \cdot)$ be an integral domain. Then it is a non-trivial commutative ring with unity having no divisor of zero. Since it is non-trivial, $n \neq 1$. We prove that n is a prime. If n be not a prime, then $n = pq$ for some integers p, q where $1 < p < n$, $1 < q < n$. So, we have $\bar{p} \in \mathbb{Z}_n$, $\bar{q} \in \mathbb{Z}_n$ and $\bar{p} \cdot \bar{q} = \overline{pq} = \bar{n} = \bar{0}$. This implies that the ring $(\mathbb{Z}_n, +, \cdot)$ contains divisors of zero, a contradiction to the hypothesis. So, n is a prime.

Conversely, let n be prime. Let $\bar{a}, \bar{b} \in \mathbb{Z}_n - \{\bar{0}\}$, i.e., $\bar{a} \neq \bar{0}$, $\bar{b} \neq \bar{0}$. Now if $\bar{a} \cdot \bar{b} = \bar{0}$ then $\overline{ab} = \bar{0}$. So, n divides ab. Since n is a prime, either n divides a or n divides b: both the cases are not possible as $\bar{a} \neq 0$, $\bar{b} \neq 0$. So, $\overline{a \cdot b} \neq \bar{0}$. So, $(\mathbb{Z}_n, +, \cdot)$ is an integral domain.

<u>Theorem 1.1.25</u> ~~A commutative ring~~ A non-trivial commutative ring R with unity is an integral domain if and only if for every non-zero element $a \in R$, $a \cdot u = a \cdot v \Rightarrow u = v$, $u, v \in R$.

Proof: Let R be a non-trivial commutative ring with unity and for a non-zero element $a \in R$, $a \cdot u = a \cdot v \Rightarrow u = v$, $u, v \in R$.

Taking $v = 0$, the condition states that

$$a \cdot u = a \cdot 0 \Rightarrow u = 0$$

$$\text{or, } a \cdot u = 0 \Rightarrow u = 0$$

This implies that a is not a left divisor of zero. Since R is commutative ring, a is not a right divisor of zero also. Thus R contains no divisor of zero and so, R is an integral domain.

Conversely, let R be an integral domain and a be a non-zero element of R.

Then $a \cdot u = a \cdot v \Rightarrow a \cdot (u - v) = 0$
$\Rightarrow u - v = 0$, (since $a \neq 0$ and R contains no divisor of zero.) $\Rightarrow u = v$

This completes the proof.