Then $p = mu$, $q = mv$ for some $u, v \in \mathbb{Z}$. Now $p - q = mu - mv$ $= m(u-v) \in S$, and for an arbitrary $r \in \mathbb{Z}$, $pr (= rp) \in S$

So, $p \in S, q \in S \Rightarrow p - q \in S$ and $p \in S, r \in \mathbb{Z} \Rightarrow rp \in S, pr \in S$.

So, S is an ideal of the ring $\mathbb{Z}$

Note 3.1.4 Since all the subrings of the ring $\mathbb{Z}$ are precisely $m\mathbb{Z}$, where m is an integer, all ideals of $\mathbb{Z}$ are given by $m\mathbb{Z}$, where m is an integer. Since the rings $m\mathbb{Z}$ and $-m\mathbb{Z}$ are identical, it follows that all ideals of the ring $\mathbb{Z}$ are given by $m\mathbb{Z}$, where m is a non-negative integer.

2. $(\mathbb{Z}, +, \cdot)$ is a subring of the ring $(\mathbb{Q}, +, \cdot)$, but $(\mathbb{Z}, +, \cdot)$ is not an ideal of $(\mathbb{Q}, +, \cdot)$ as $3 \in \mathbb{Z}$, $\frac{1}{2} \in \mathbb{Q}$ but $3 \cdot \frac{1}{2} = \frac{3}{2} \notin \mathbb{Z}$

3. Let R be the ring of all real valued continuous functions defined on $[0,1]$ and let $S = \{f \in R : f(\frac{1}{2}) = 0\}$

Then S is an ideal of R. S is a non-empty set as zero function $0 \in S$. Let $f, g \in S$. Then f, g are continuous on $[0,1]$ and $f(\frac{1}{2}) = 0$, $g(\frac{1}{2}) = 0$.

So, $f - g$ is continuous on $[0,1]$ and $(f-g)(\frac{1}{2}) = f(\frac{1}{2}) - g(\frac{1}{2}) = 0$

This shows that $f - g \in S$. Let $h \in R$. Then h is continuous on $[0,1]$. If $f \in S$, then hf and fh are both continuous on $[0,1]$ and $(hf)(\frac{1}{2}) = h(\frac{1}{2}) f(\frac{1}{2}) = 0$. Similarly, $(fh)(\frac{1}{2}) = f(\frac{1}{2}) h(\frac{1}{2}) = 0$. So, $hf \in S$ and $fh \in S$. So, $f \in S, h \in R \Rightarrow hf \in S$ and $fh \in S$. So, S is an ideal of R.

4. Let $R = M_2(\mathbb{Z})$ and $S = M_2(2\mathbb{Z})$. Then S is an ideal of R. Let $A = \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix} \in S$, $B = \begin{pmatrix} 2p & 2q \\ 2r & 2s \end{pmatrix} \in S$

Then $A-B = \begin{pmatrix} 2(a-p) & 2(b-q) \\ 2(c-r) & 2(d-s) \end{pmatrix} \in S$. Let $P = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ be an arbitrary member of $M_2(\mathbb{Z})$ and let $A = \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix} \in M_2(2\mathbb{Z})$

Then $AP = \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 2(ap+br) & 2(aq+bs) \\ 2(cp+dr) & 2(cq+ds) \end{pmatrix} \in S$ and

$$PA = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix} = \begin{pmatrix} 2(pa+qc) & 2(pb+qd) \\ 2(ra+sc) & 2(rb+sd) \end{pmatrix} \in S$$

So, S is an ideal of R.

Theorem 3.1.5 Let R be a ring and $\{I_\alpha : \alpha \in A\}$ be a collection of ideals of R (A is the index set). Then $I = \bigcap_{\alpha \in A} I_\alpha$ is an ideal of R.

Proof: Let $\{I_\alpha : \alpha \in A\}$ be a collection of ideals of R. Since, each I_α is a subring of R, by Theorem 2.1.8, I is a subring of R. Let $x \in I$ and $r \in R$. Then $x \in I_\alpha$ for each $\alpha \in A$ and hence $xr \in I_\alpha$ for each $\alpha \in A$ as I_α is an ideal for each $\alpha \in A$. So, $xr \in I = \bigcap_{\alpha \in A} I_\alpha$

~~So, I is an~~ Similarly if $x \in I$ and $r \in R$, then $rx \in I$. So, I is an ideal of R.

Note 3.1.6: Unions of two ideals may not be an ideal. $2\mathbb{Z}$ and $3\mathbb{Z}$ are ideals of $\mathbb{Z}$ but $2\mathbb{Z} \cup 3\mathbb{Z}$ is not an ideal of $\mathbb{Z}$.

Definition Let R be a ring. Let I and J be two ideals of R.

Define $I+J = \{a+b : a \in I, b \in J\}$ and

$$IJ = \left\{ \sum_{i=1}^{n} a_i b_i : a_i \in I, b_i \in J, i = 1, 2, \ldots, n, n \in \mathbb{N} \right\}$$

Theorem 3.1.7 Let R be a ring and I, J be two ideals of R. Then $I+J$ and IJ are ideals of R. Moreover, $IJ \subseteq I \cap J$ and $I \cup J \subseteq I+J$. Indeed $I+J$ is the smallest ideal containing $I \cup J$.

Proof: Let $x_1, x_2 \in I+J$. Then $x_1 = a_1 + b_1$, $x_2 = a_2 + b_2$, $a_1, a_2 \in I$ and $b_1, b_2 \in J$. Now $x_1 - x_2 = (a_1 + b_1) - (a_2 + b_2) = (a_1 - a_2) + b_1 - b_2 \in I+J$ as $a_1 - a_2 \in I$ and $b_1 - b_2 \in J$. Now for any $r \in R$ $rx_1 = r(a_1 + b_1) = ra_1 + rb_1 \in I+J$ as $ra_1 \in I$ and $rb_1 \in J$. Similarly $x_1 r = a_1 r + b_1 r \in I+J$. So, $I+J$ is an ideal of R.

Let $x, y \in IJ$ where $x = \sum_{i=1}^{n} a_i b_i$, $a_i \in I$, $b_i \in J$, $i = 1, 2, \ldots n$ and $y = \sum_{i=1}^{m} c_i d_i$, $c_i \in I$, $d_i \in J$, $i = 1, 2, \ldots, m$, $n, m \in \mathbb{N}$

Then $x - y = \sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{m} c_i d_i = \sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{m} (-c_i) d_i$

where $a_i, -c_i \in I$, $i = 1, 2, \ldots, n$ and $b_i, d_i \in J$, $i = 1, 2, \ldots, m$, $n, m \in \mathbb{N}$

So, $x - y \in IJ$

Similarly, for any $r \in R$ $rx = r \sum_{i=1}^{m} a_i b_i = \sum_{i=1}^{n} (ra_i) b_i \in IJ$ as $ra_i \in I$, $i = 1, 2, \ldots, n$ and $b_i \in J$, $i = 1, 2, \ldots, m$. Similarly, $xr \in IJ$

Hence IJ is an ideal of R.

Now since I and J are ideals of R $IJ \subseteq I \cap J$. Again for any element $a \in I$, $a = a + 0 \in I+J$. So, $I \subset I+J$. Similarly, $J \subseteq I+J$. So, $I \cup J \subseteq I+J$. Finally let K be any ideal of R containing both I and J. ~~Then obviously~~ Let $x \in I+J$ then $x = a+b$, $a \in I$ and $b \in J \Rightarrow a, b \in K$ $\Rightarrow a+b \in K \Rightarrow x \in K$. So, $I+J \subseteq K$, proving $I+J$ is the smallest ideal containing ~~I and J~~. $I \cup J$.

Theorem 3.1.8 If an ideal S of a ring R with unity contains a unit of R, then $S = R$.

Proof: Let R be a ring with unity 1 and let S contains a unit u of R. Since u is a unit, $u^{-1} \in R$ and $uu^{-1} = u^{-1}u = 1$.

Since S is an ideal of R $u \in S$, $u^{-1} \in R \Rightarrow uu^{-1} \in S \Rightarrow 1 \in S$.

Let $a \in R$. Since S is an ideal of R, $a \in R$, $1 \in S \Rightarrow a \in S$

So, $a \in R \Rightarrow a \in S$. So, $R = S$.

Corollary 3.1.9. A division ring has no non-trivial proper ideals.

Proof: Let R be a division. Then it is a ring with unity and every non-zero element of R is a unit.

Let S be a non-trivial ideal of R. Then S contains a non-zero element of R which is also a unit in R. Since S is an ideal of a ring R with unity and S contains a unit of R, So, from Theorem 3.1.8, $S = R$.

So, S has no non-trivial proper ideals.

Corollary 3.1.10 A field has no non-trivial proper ideal.

Proof: As a field is a division ring, so has no non-trivial proper ideal.

Definition 3.1.11 (Simple ring) A ring is said to be a simple ring if it has no nontrivial proper ideals.

Examples: A division ring is a simple ring. A field is a simple ring.

Theorem 3.1.12 A commutative ring with unity having no non-trivial proper ideals is a field. i.e., A simple commutative ring with unity is a field.

Proof: Let R be a commutative ring with unity 1 and the