the only ideals of R are the null ideal and R itself. Let a be a non-zero element in R. Let us consider the set $Ra = \{ra : r \in R\}$. Let $u, v \in Ra$ Then $u = r_1 a$ and $v = r_2 a$ for some $r_1, r_2 \in R$. $u - v = r_1 a - r_2 a = (r_1 - r_2) a \in Ra$

For an arbitrary r in R, $ru = r(r_1 a) = (rr_1) a \in Ra$, since $rr_1 \in R$

So, $u \in Ra, v \in Ra \Rightarrow u - v \in Ra$ and $u \in Ra, r \in R \Rightarrow ru \in Ra$. So, Ra is is a left ideal of R. Since R is commutative, Ra is an ideal of R. Since $1 \in R$, $1a \in R$. So, $a \in Ra$.

Thus Ra is a non-null ideal of R. Since R has no proper ideal $Ra = R$. Since $1 \in R$, $1 \in Ra$ and therefore $1 = ba$ for some $b \in R$. This proves that a is a unit in R. So, every non-zero element is a unit. Consequently, R is a field. This completes the proof.

<u>Note 3.1.13</u> A commutative ring (without units) having no non-trivial proper ideals may not be a field.

Let $(R, +)$ be a cyclic group of prime order.

Let us define multiplication in R by $a \cdot b = 0$ for all $a, b \in R$. Then $(R, +, \cdot)$ is a commutative ring without unity. Since a cyclic group of prime order has no non-trivial proper subgroups, the ring has no non-trivial proper ideals. It is not a field, since it contains divisors of zero.

Note 3.1.14 A simple non-commutative ring with unity may not be a division ring.

The ring $M_2(\mathbb{R})$ is a non-commutative ring with unity

It is not a division ring. We show that $M_2(\mathbb{R})$ is a simple ring. Let U be a non-zero ideal of the ring $M_2(\mathbb{R})$. Let $A=(a_{ij})$ be a non-zero matrix in U. Since $A \neq 0$, at least one a_{ij}, say $a_{rs} \neq 0$.

Let E_{ij} be the 2×2 matrix whose ijth element is 1 and every other element is 0. $E_{ij} \in M_2(\mathbb{R})$ for $i=1,2$, $j=1,2$

We have $E_{ij} A E_{kl} = a_{jk} E_{il}$ for all admissible i,j,k,l.
(It is a result of Matrix theory)

Since U is an ideal, $E_{1r} A E_{s1} \in U$, i.e., $a_{rs} E_{11} \in U$; also $E_{2r} A E_{s2} \in U$, i.e., $a_{rs} E_{22} \in U$.

But $a_{rs} E_{11} = \begin{pmatrix} a_{rs} & 0 \\ 0 & 0 \end{pmatrix}$ and $a_{rs} E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & a_{rs} \end{pmatrix}$

So, $\begin{pmatrix} a_{rs} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & a_{rs} \end{pmatrix} \in U$ or, $P = \begin{pmatrix} a_{rs} & 0 \\ 0 & a_{rs} \end{pmatrix} \in U$

Since $a_{rs} \neq 0$, $a_{rs}^{-1} \in \mathbb{R}$ and $Q = \begin{pmatrix} a_{rs}^{-1} & 0 \\ 0 & a_{rs}^{-1} \end{pmatrix} \in M_2(\mathbb{R})$

Since U is an ideal $PQ \in U$, i.e., $I_2 \in U$

Let B be an arbitrary element in $M_2(\mathbb{R})$. Then $B \cdot I_2 \in U$ or, $B \in U$. So, $B \in M_2(\mathbb{R}) \Rightarrow B \in U$.

So, $M_2(\mathbb{R}) = U$. This proves that $M_2(\mathbb{R})$ has no proper non-trivial ideal.

Example: Let p and q be positive integers prime to each other. Then the ideal $p\mathbb{Z}+q\mathbb{Z}$ of the ring $\mathbb{Z}$ is the ~~ideal~~ ideal $\mathbb{Z}$ itself.

Since p, q are prime to each other, $pu + qv = 1$ for some $u, v \in \mathbb{Z}$. Since $u \in \mathbb{Z}$, $pu \in p\mathbb{Z}$, since $v \in \mathbb{Z}$ $qv \in q\mathbb{Z}$

So, $pu + qv \in p\mathbb{Z} + q\mathbb{Z}$. So, $1 \in p\mathbb{Z} + q\mathbb{Z}$

Let r be an element in the ring $\mathbb{Z}$

$1 \in p\mathbb{Z} + q\mathbb{Z}$, $r \in \mathbb{Z} \Rightarrow r \in p\mathbb{Z} + q\mathbb{Z}$. So,

$$p\mathbb{Z} + q\mathbb{Z} = \mathbb{Z}$$

Definition 3.1.14 Let S be a non-empty subset of a ring R. The intersection of all ideals of R containing the subset S is an ideal of R and it is the smallest ideal of R containing the subset S. This ideal is said to be the ideal generated by S.

In particular, if S be a single element of R then the ideal generated by the element is called a principal ideal of R. Thus corresponding to each element of a ring there is a principal ideal of R. i.e., if R be a ring and $a \in R$, the smallest ideal of R containing the element a is said to be the principal ideal generated by a and is denoted by $\langle a \rangle$. Alternatively, an ideal U of a ring R is said to be a principal ideal of R if $U = \langle a \rangle$ for some $a \in R$.

Examples 1. Let R be a ring. The null ideal $\{0\}$ is the smallest ideal of R containing the element 0. The null ideal $\{0\}$ is a principal ideal of R.

2. Let R be a ring with unity. Let 1 be the unity in R. Let U be the smallest ideal of R containing the element 1. Since U is an ideal, $a \in R$, $1 \in U \Rightarrow 1 \cdot a \in U \Rightarrow a \in U$.
So, $U = R$. Consequently R is the principal ideal generated by 1.

3. In the ring $\mathbb{Z}$, the ring of all integers, the subring $m\mathbb{Z}$ (m being a positive integer) is a principal ideal.
The subring $m\mathbb{Z}$ is an ideal of the ring $\mathbb{Z}$. We prove that it is the smallest ideal containing the element m. Let U be an ideal of $\mathbb{Z}$ containing the element m. Then $ma \in U$ for all $a \in \mathbb{Z}$. In other words, $m\mathbb{Z} \subseteq U$. This proves that $m\mathbb{Z}$ is the smallest ideal containing m. So, $m\mathbb{Z}$ is a principal ideal of the ring $\mathbb{Z}$.

Note: Since all ideals in the ring $\mathbb{Z}$ are given by $m\mathbb{Z}$, where m is an integer and $m \geq 0$, it follows that all ideals of the ring $\mathbb{Z}$ are principal ideals.

4. In the ring $2\mathbb{Z}$, the subring $4\mathbb{Z}$ is an ideal. It is the smallest ideal containing the element 4. So, it is the principal ideal $\langle 4 \rangle$.

5. Let us consider the ring $R = \mathbb{Z} \times \mathbb{Z}$, where $+$ and $\cdot$ are defined by $(a,b)+(c,d) = (a+c, b+d)$ and $(a,b)\cdot(c,d) = (ac, bd)$ for all $(a,b),(c,d) \in \mathbb{Z}\times\mathbb{Z}$. The subring $S = \{(a,0) : a \in \mathbb{Z}\}$ is an ideal of R. We prove that it is the smallest ideal of R containing the element $(1,0)$ of R.