Theorem 3.2.1 Let R be a ring and I be an ideal of R

(i) If J is an ideal of R such that $I \subseteq J \subseteq R$, then J/I is an ideal of R/I

(ii) Every ideal of R/I is equal to M/I for some ideal M of R such that $I \subseteq M \subseteq R$.

Proof: (i) $I \subseteq J \subseteq R$. Since I is an ideal of R, $a, b \in I$ $\Rightarrow a-b \in I$ and $a \in I, r \in R \Rightarrow ra \in I$ and $ar \in I$

So, $a, b \in I \Rightarrow a-b$ and $a \in I$, $s \in J \Rightarrow sa \in I$ and $as \in I$, since $s \in R$ also. This proves that I is an ideal of J and so, quotient ring J/I exists.

Since $I \subseteq J \subseteq R$, J/I is a non-empty subset R/I

Let $a+I \in J/I$ and $b+I \in J/I$. Then $a \in J$, $b \in J$

Since J is an ideal of R, $a-b \in J$ and $a \in J$, $r \in R$ $\Rightarrow ra \in J$ and $ar \in J$.

So, $(a-b)+I \in J/I$ and $ra+I = (r+I)(a+I) \in J/I$. Also $ar+I = (a+I)(r+I) \in J/I$. This proves that J/I is an ideal of R/I

(ii) Let T be an ideal of R/I. Let M be a subset of R defined by $M = \bigcup_{a+I \in T} \{a\}$. Let $a+I \in T$ and $b+I \in T$. Then $a, b \in M$. Since T is an ideal of R/I, $a+I \in T$, $b+I \in T$ $\Rightarrow (a+I)-(b+I) \in T$ and $r+I \in R/I \Rightarrow (a+I)(r+I) \in T$ and $(r+I)(a+I) \in T$. This shows that $a-b+I \in T \Rightarrow a-b \in M$

Also $ar+I \in T \Rightarrow ar \in M$ and $ra+I \in T \Rightarrow ra \in M$

So, we see that if $a, b \in M$ then $a-b \in M$ and if $a \in M$ and $r \in R$ then $ar \in M$ and $ra \in M$. So, M is an ideal of R containing I. Since I is an ideal of R and M is an ideal of R containing I, I is an ideal of M. So, the quotient ring M/I exists and $T = M/I$

Theorem 3.2.2 In a commutative ring R with unity, an ideal P is a prime ideal if and only if the quotient ring R/P is an integral domain..

Proof: Let P be a prime ideal in ~~the ring~~ the ring R. Since R is a commutative ring with unity, the quotient ring R/P is a commutative ring with unity.

Let $(a+P)(b+P) = 0$ where $a+P, b+P \in R/P$. Then $a, b \in R$

$(a+P)(b+P) = (ab+P) = 0 \Rightarrow ab \in P$ (As zero element 0 of R/P is P)

Since P is a prime ideal, $ab \in P$ implies either $a \in P$ or $b \in P$. So, either $a+P = 0$ or $b+P = 0$

So, the quotient ring R/P contains no divisor of zero. So, it is an integral domain.

Conversely, let P be an ideal in the ring R and the quotient ring R/P is an integral domain.

Let $ab \in P$. Then $ab+P = P$. So, $(a+P)(b+P) = P$.

But P is the zero element of R/P. Since it is an integral domain. So, we have, either

$a+P = P$ or $b+P = P \Rightarrow$ either $a \in P$ or $b \in P$

So, $ab \in P \Rightarrow$ either $a \in P$ or $b \in P$. Hence P is a prime ideal of R. This completes the proof

Theorem 3.2.3 In a commutative ring with unity, an ideal M is a maximal ideal if and only if the quotient ring R/M is a field.

Proof: Let 1 be the unity in R. Let M be a maximal ideal. Then $M \neq R$ and so $1 \notin M$ and the quotient ring R/M is non-trivial. Since R is a commutative ring with unity 1, the quotient ring R/M is a commutative ring with unity $1+M$. Let $a+M$ be a non-zero element of the ring R/M. Then $a \notin M$. Let T be the principal of R generated by a. Since R is a commutative ring with unity, $T = aR = \{ar : r \in R\}$

$M+T$ is an ideal of R. $a \in T \Rightarrow a \in M+T$. Since $a \notin M$, the ideal M is strictly contained in the ideal $M+T$. But M is a maximal ideal of R and so, $M+T = R$

$1 \in R \Rightarrow 1 \in M+T \Rightarrow 1 = b + ar$ for some $b \in M$, some $r \in R$

$$1+M = (b+ar)+M = (b+M)+(ar+M) = M + (ar+M)$$
$$= ar+M = (a+M)(r+M)$$

Since R/M is a commutative ring, $(a+M)(r+M) = (r+M)(a+M)$
$= 1+M$

This shows that $a+M$ is a unit. So, every non-zero element in ring R/M is a unit and so, R/M is a field.

Conversely, let R/M be a field. Since a field is a non-trivial ring, $M \neq R$. So 1, the unity in R does not belong to M. $1+M$ is the unity element in the ring R/M. Let U be an ideal of R such that M is a proper subset of U. Let us choose an element $a \in U$ but $a \notin M$. Then $a+M$ is a non-zero element in the ring R/M. Since R/M is a field, $a+M$ is invertible and therefore there exists a non-zero element $b+M$ in R/M such that

$(a+M)(b+M) = 1+M$. This implies $ab+M = 1+M$

So, $ab-1 \in M$

Now $a \in U \Rightarrow ab \in U$, since U is an ideal and

$ab-1 \in M \Rightarrow ab-1 \in U$, since M is a proper subset of U. So, $ab-(ab-1) \in U$, i.e., $1 \in U$. So, $U=R$.

Thus M is a proper subset of $U \Rightarrow U = R$. Hence M is maximal. This completes the proof.

Theorem 3.2.4 Every maximal ideal in a commutative ring with unity is a prime ideal.

Proof: Let M be a maximal ideal in a commutative ring R with with unity. Then by Theorem 3.2.3, R/M is a field and hence an integral domain. So, by Theorem 3.2.2 M is a prime ideal.

Note 3.2.5 A prime ideal in a commutative ring R with unity may not be a maximal ideal. For example, the null ideal is a prime ideal in the ring $\mathbb{Z}$ but it is not a maximal ideal in the ring $\mathbb{Z}$.

Note 3.2.6 A maximal ideal in a commutative ring (without unity) may not be a prime ideal

Let consider the ideal $4\mathbb{Z}$ in the ring $2\mathbb{Z}$.