Let U be an ideal of the ring $2\mathbb{Z}$ properly containing $4\mathbb{Z}$. Then $\exists$ an element $p \in U$ such that $p \notin 4\mathbb{Z}$. Then $p = 4m + 2$ where m is an integer. $4m \in 4\mathbb{Z}$ and therefore $4m \in U$
$\Rightarrow p - 4m \in U$ as $p \in U$ and $4m \in U$ and U is an ideal.
i.e., $2 \in U$ and this again implies $U = 2\mathbb{Z}$. ~~This proves that $4\mathbb{Z}$ is maximal~~ So, ~~$4\mathbb{Z} \subset$~~ $4\mathbb{Z} \subset U \subset 2\mathbb{Z}$ implies $U = 2\mathbb{Z}$. So, this proves that $4\mathbb{Z}$ is maximal.
The ideal $4\mathbb{Z}$ in the ring $2\mathbb{Z}$ is not prime as $2.2 \in 4\mathbb{Z}$ but $2 \notin 4\mathbb{Z}$. So, $4\mathbb{Z}$ is a maximal ideal but not a prime ideal in the ring $2\mathbb{Z}$

<u>Worked examples</u> 1. Find the maximal ideals and the prime ideals of the ring $\mathbb{Z}_6$

Solution: The ring $\mathbb{Z}_6$ is a principal ideal ring.
The ideals are $\langle \bar{0} \rangle = \{\bar{0}\}$, $\langle \bar{1} \rangle = \mathbb{Z}_6$, $\langle \bar{2} \rangle = \{\bar{0}, \bar{2}, \bar{4}\}$
$\langle \bar{3} \rangle = \{\bar{0}, \bar{3}\}$ $\langle \bar{4} \rangle = \langle \bar{2} \rangle$, $\langle \bar{5} \rangle = \mathbb{Z}_6$. So,
$\langle \bar{0} \rangle \subset \langle \bar{4} \rangle = \langle \bar{2} \rangle \subset \mathbb{Z}_6$, $\langle \bar{0} \rangle \subset \langle \bar{3} \rangle \subset \mathbb{Z}_6$
The maximal ideals of the ring are $\langle \bar{2} \rangle = \langle \bar{4} \rangle$ and $\langle \bar{3} \rangle$
Since $\mathbb{Z}_6$ is a commutative ring with ~~unit~~ unity, every maximal ideal of the ring is a prime ideal. Then let P be a prime ideal of $R = \mathbb{Z}_6$. Then the quotient ring R/P is an integral domain. Since $R = \mathbb{Z}_6$ is finite, the ring R/P is finite. Hence R/P is a field and therefore the ideal P is maximal. Hence the prime ideals of the ring $\mathbb{Z}_6$ are $\langle \bar{2} \rangle = \langle \bar{4} \rangle$ and $\langle \bar{3} \rangle$.

3.3 Homomorphism of rings

Definition 3.3.1 Let R and R' be two rings. A mapping $\phi : R \to R'$ is said to be a homomorphism if

(i) $\phi(a+b) = \phi(a) + \phi(b)$, and

(ii) $\phi(ab) = \phi(a)\phi(b)$

A ring homomorphism $\phi : R \to R'$ is called

(i) a monomorphism, if ϕ is injective

(ii) an epimorphism, if ϕ is surjective

(iii) an isomorphism, if ϕ is bijective.

When ϕ is an isomorphism, $\phi^{-1} : R' \to R$ is also an isomorphism. In this case R and R' are said to be isomorphic rings and we write $R \simeq R'$

A ring homomorphism ϕ is, in particular, a group homomorphism from $(R, +)$ to $(R', +)$. So, $\phi(0_R) = 0_{R'}$, $\phi(-a) = -\phi(a)$, $a \in R$, 0_R and $0_{R'}$ are zero elements of R and R' respectively. But since $(R - \{0_R\}, \cdot)$ and $(R' - \{0_{R'}\})$ are not always groups, ϕ does not inherit properties of a multiplicative group homomorphism. It may happen that R and R' are both rings with units but $\phi(I_R) \neq \phi(I_{R'})$, I_R and $I_{R'}$ are the ~~units~~ multiplicative identities of R and R'.

Examples 1. Let R and R' be two rings and $\phi : R \to R'$ be defined by $\phi(r) = 0'$ for all $r \in R$. $0'$ being the zero element in R'.

Let $a, b \in R$. Then $a+b \in R$ and $a \cdot b \in R$

By the condition, $\phi(a) = 0'$, $\phi(b) = 0'$. $\phi(a+b) = 0'$, $\phi(a \cdot b) = 0'$

~~for all r∈R, 0′ being the zero element in R′.~~

Now $\phi(a+b) = \phi(a) + \phi(b)$ and $\phi(a\cdot b) = \phi(a)\cdot\phi(b)$ for all $a, b \in R$.

So, ϕ is a homomorphism.

Note: This homomorphism is called the ~~zero~~ trivial ring homomorphism.

2. Let $R = (\mathbb{Z}, +, \cdot)$ and $\phi : R \to R$ be defined by $\phi(x) = 2x$, $x \in \mathbb{Z}$.

Let $a, b \in \mathbb{Z}$. Then $a\cdot b \in \mathbb{Z}$ and $\phi(a\cdot b) = 2ab$

But $\phi(a)\cdot\phi(b) = 2a\cdot 2b = 4ab \neq \phi(a)\cdot\phi(b)$

Therefore ϕ is not a homomorphism.

3. Let $R = (\mathbb{Z}, +, \cdot)$ and $\phi : R \to R$ be defined by $\phi(x) = -x$, $x \in \mathbb{Z}$. Let $a, b \in \mathbb{Z}$. Then $a\cdot b \in \mathbb{Z}$ and $\phi(a\cdot b) = -ab$.

But $\phi(a)\cdot\phi(b) = -a\cdot(-b) = ab \neq \phi(a)\cdot\phi(b)$

So, ϕ is not a homomorphism.

4. Let $R = (\mathbb{C}, +, \cdot)$ and $\phi : R \to R$ be defined by $\phi(z) = \bar{z}$, $z \in \mathbb{C}$ and $\bar{z}$ is the conjugate of the complex number z.

Let $z_1, z_2 \in \mathbb{C}$. Then $z_1 + z_2 \in \mathbb{C}$, $z_1 \cdot z_2 \in \mathbb{C}$.

$\phi(z_1) = \bar{z_1}$, $\phi(z_2) = \bar{z_2}$, $\phi(z_1+z_2) = \overline{z_1+z_2}$, $\phi(z_1\cdot z_2) = \overline{z_1\cdot z_2}$.

We have $\phi(z_1+z_2) = \overline{z_1+z_2} = \bar{z_1} + \bar{z_2} = \phi(z_1) + \phi(z_2)$ and

$\phi(z_1\cdot z_2) = \overline{z_1\cdot z_2} = \bar{z_1}\cdot\bar{z_2} = \phi(z_1)\cdot\phi(z_2)$

So, ϕ is a homomorphism.

ϕ is a bijection. Hence ϕ is an isomorphism.

Theorem 3.3.2 Let R and R' be two rings and $\phi : R \to R'$ be a homomorphism. Then

(i) $\phi(0) = 0'$, where 0 is the zero element in R and $0'$ is that in R'.

(ii) $\phi(-a) = -\phi(a)$ for all $a \in R$.

Proof: (i) $0 = 0 + 0$ in R.

So, $\phi(0) = \phi(0+0)$ in R'

$= \phi(0) + \phi(0)$, since ϕ is a homomorphism.

Since $0'$ is the zero element in R', $\phi(0) + 0' = \phi(0) + \phi(0)$.
So, $0' = \phi(0)$, by left cancellation law for addition.

(ii) Let $a \in R$. Then $a + (-a) = (-a) + a = 0$ in R.
We have $\phi(0) = \phi(a + (-a)) = \phi(a) + \phi(-a)$ and also
$\phi(0) = \phi((-a) + a) = \phi(-a) + \phi(a)$
So, $\phi(a) + \phi(-a) = \phi(-a) + \phi(a) = 0'$
This shows that $\phi(-a)$ is the additive inverse of $\phi(a)$ in R'.
That is, $\phi(-a) = -\phi(a)$

Theorem 3.3.3 Let R, R' be two rings and $\phi : R \to R'$ be an onto homomorphism. Then
(i) If R has a unity ~~is~~ 1, then R' has $\phi(1)$ as unity
(ii) If a be a unit in R, then $\phi(a)$ is a unit in R' and $(\phi(a))^{-1} = \phi(a^{-1})$

Proof: (i) Let $a' \in R'$. Since ϕ is onto, $\exists$ $a \in R$ such that $\phi(a) = a'$. We have $a \cdot 1 = 1 \cdot a = a$ in R
Since ϕ is a homomorphism $\phi(a) \cdot \phi(1) = \phi(1) \cdot \phi(a) = \phi(a)$.
That is, $a' \cdot \phi(1) = \phi(1) \cdot a' = a'$ for all a' in R'
This shows that $\phi(1)$ is the unity in R'.

(ii) Since a is a unit in R, R is a ring with unity 1 (say). Since a is a unit in R, $a^{-1} \in R$, and $a \cdot a^{-1} = a^{-1} \cdot a = 1$ holds in R. ~~Since~~ Since ϕ is a homomorphism $\phi(a) \cdot \phi(a^{-1}) = \phi(a^{-1}) \cdot \phi(a) = \phi(1)$
Since ϕ is onto, $\phi(1)$ is the unity in R' and it follows that $\phi(a)$ is a unit in R and $(\phi(a))^{-1} = \phi(a^{-1})$