In particular, V is said to be a real vector space (or a complex vector space) if the field F be the field of real numbers $\mathbb{R}$ (or if ~~the field~~ the field of complex numbers $\mathbb{C}$).

Example 1 The set of all n-tuples with entries from field F is denoted by F^n. This set is a vector space over the field F with the operations of coordinatewise addition and scalar multiplication; that is, if $u=(a_1,a_2,\ldots,a_n)\in F^n$, $v=(b_1,b_2,\ldots,b_n)\in F^n$, and $c\in F$, then

$$u+v=(a_1+b_1,a_2+b_2,\ldots,a_n+b_n) \text{ and } cu=(ca_1,ca_2,\ldots,ca_n).$$

So, $\mathbb{R}^3$ is a real vector space. In this vector space,

$$(3,-2,0)+(-1,1,4)=(2,-1,4) \text{ and } -5(1,-2,0)=(-5,10,0)$$

Similarly $\mathbb{C}^2$ is a complex vector space. In this vector space $(1+i,2)+(2+3i,1-5i)=(3+4i,3-5i)$

and $i(1+i,2-i)=(-1+i,1+2i)$

Sometimes, vectors in F^n, may be written as column vectors $\begin{pmatrix} a_1\\ a_2\\ \vdots\\ a_n\end{pmatrix}$. Since 1-tuple whose only entry is from F can be regarded as an element F, we usually write F rather than F^1 for the vector space of 1-tuple with entry from F.

Example 2 The set of all $m\times n$ matrices with entries from a field F is a vector space, denoted by $M_{m\times n}(F)$, with the matrix addition as vector addition and multiplication of a matrix by a scalar as scalar multiplication; That is, if $A=(a_{ij})_{m\times n}\in M_{m\times n}(F)$ and $B=(b_{ij})_{m\times n}\in M_{m\times n}(F)$ and $c\in F$

Then $A+B=(c_{ij})_{m\times n}$ where $c_{ij}=a_{ij}+b_{ij}$, $i=1,\ldots,m$, $j=1,2,\ldots,n$

and $cA = [d_{ij}]_{m\times n}$ where $d_{ij} = ca_{ij}$ $i=1,2,\ldots,m$, $j=1,2,\ldots,n$

Example 3 Let S be a non-empty set and F be any field, and let $\mathcal{B}(S,F)$ denote the set of all functions from S to F. The set $\mathcal{B}(S,F)$ is a vector space over the field F with the operations of addition and scalar multiplication defined for $f, g \in \mathcal{B}(S,F)$ and $c \in F$ by

$$(f+g)(x) = f(x)+g(x) \text{ for all } x \in S$$

and $$(cf)(x) = c\big(f(x)\big) \text{ for } x \in S$$

Example 4. Let F be any F. Let $P(F)$ be the set of all polynomials with coefficients from F. Let $f(x), g(x) \in P(F)$

and $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

and $g(x) = b_m x^m + a_{m-1}x^{m-1} + \cdots + b_1 x + b_0$, $m \le n$

Define $f(x)+g(x) = (a_n+b_n)x^n + (a_{n-1}+b_{n-1})x^{n-1} + \cdots + (a_1+b_1)x + (a_0+b_0)$

where we define $b_{m+1} = b_{m+2} = \cdots = b_n = 0$

and $cf(x) = ca_n x^n + ca_{n-1}x^{n-1} + \cdots + ca_1 x + ca_0$

With these two operations of addition and scalar multiplication, the set $P(F)$ forms a vector space over the field F.

(Note: Here the zero polynomial $f(x)=0$. Its degree, for convenience, taken as -1 and if $f(x)=c$, c is a non-zero constant then degree of $f(x)$ is zero.)

Theorem 2.1.1 In a vector space V over a field F,

(i) $0\alpha = \theta$ for all $\alpha \in V$

(ii) $c\theta = \theta$ for all $c \in F$

(iii) $(-1)\alpha = -\alpha$ for all $\alpha \in V$, 1 being the multiplicative identity in F.

(iv) $c\alpha = \theta$ implies either $c=0$ or $\alpha = \theta$.

Proof:
(i) 0 is the zero element in F. So,

$0+0=0$ in F

$\Rightarrow (0+0)\alpha = 0\alpha$ for any $\alpha \in V$

$\Rightarrow 0\alpha + 0\alpha = 0\alpha$ by V9

$-0\alpha \in V$ since $0\alpha \in V$

So, $-0\alpha + (0\alpha + 0\alpha) = -0\alpha + 0\alpha$

or, $(-0\alpha + 0\alpha) + 0\alpha = \theta$ by ~~V3~~ V3 and V5

or, $\theta + 0\alpha = \theta$ by V5

or, $0\alpha = \theta$ by V4

(ii) θ is the zero element in V. So,

$\theta + \theta = \theta$ in V

$\Rightarrow c(\theta+\theta) = c\theta$

$\Rightarrow c\theta + c\theta = c\theta$, by ~~V8~~ V8

$-c\theta \in V$ as $c\theta \in V$. So,

$-c\theta + (c\theta + c\theta) = -c\theta + c\theta$

or, $(-c\theta + c\theta) + c\theta = \theta$ by V3 and V5

or, $\theta + c\theta = \theta$ by V5

or, $c\theta = \theta$ by V4

(iii) We have $\theta = 0\alpha$ by (i)

$= [1+(-1)]\alpha$

$= 1\alpha + (-1)\alpha$ by V9

$= \alpha + (-1)\alpha$ by V10

So, $-\alpha + \theta = -\alpha + (\alpha + (-1)\alpha)$

or, $-\alpha + \theta = (-\alpha + \alpha) + (-1)\alpha$ by V3

or, $-\alpha = \theta + (-1)\alpha$ by V5, V4

or, $-\alpha = (-1)\alpha$ by V4

So, $(-1)\alpha = -\alpha$

(iv) Let $c\alpha = \theta$ and let $c \neq 0$. Then c^{-1} exists in F

Now $c\alpha = \theta \Rightarrow c^{-1}(c\alpha) = c^{-1}\theta$

$\Rightarrow (c^{-1}c)\alpha = c^{-1}\theta$ by V7

$\Rightarrow 1\alpha = \theta$ by (ii)

$\Rightarrow \alpha = \theta$ by V10

So, $c\alpha = \theta$ and $c \neq 0 \Rightarrow \alpha = \theta$

Contrapositively, $c\alpha = \theta$ and $\alpha \neq \theta \Rightarrow c = 0$

Hence $c\alpha = \theta$ implies either $c = 0$ or $\alpha = \theta$

Definition 2.1.2 (Subspace)

Let V be a vector space over the field F. Let W be a non-empty subset of V. W is said to be a subspace of V if W forms a vector space with respect to the same operation of addition and scalar multiplication as on V restricted to W and $F \times W$.

Examples: 1. Let V be a vector space over a field F. Then V itself is a subspace of V. This subspace is called the improper subspace of V.

The set $\{\theta\}$ consisting only the zero element of V (called the null vector) forms a subspace of V. It is called the trivial subspace of V.

2. Let $S = \{(x,y) \in \mathbb{R}^2 : y = 2x\}$. S is a subset of $\mathbb{R}^2$ the real vector space $\mathbb{R}^2$. Then S is non-empty as $(0,0) \in S$. S is a subspace of $\mathbb{R}^2$.

3. Let $S = \{(x,y,z) \in \mathbb{R}^3 : x = 0\}$. S is a non-empty subset of $\mathbb{R}^3$ as $(0,0,0) \in S$. S is a subspace of $\mathbb{R}^3$.