Note: If $\phi: R \to R'$ be an onto ~~homo~~ homomorphism and R' be a ring with unity, then R may not be a ring with unity.

For example, let R be ring of 2×2 matrices over $\mathbb{Z}$ of the form $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$, $R' = \mathbb{Z}$ and $\phi: R \to R'$ be defined by $\phi\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right) = a$. Then ϕ is ~~an~~ a ring epimorphism. Here R' is a ring with unity but R is not.

Definition 3.3.4 Let R and R' be two rings and $\phi: R \to R'$ be a homomorphism. Then the set $\phi(R) = \{\phi(a) : a \in R\}$ is a subset of R'. It is said to be the homomorphic image of R under ϕ.

Theorem 3.3.5 Let R and R' be two rings and $\phi: R \to R'$ be a ring homomorphism. Then $\phi(R)$ is a subring of R'.

Proof: $\phi(R)$ is a non-empty ~~set~~ subset of R' as $0' = \phi(0) \in \phi(R)$. Let $a', b' \in \phi(R)$. So, $\exists\ a, b \in R$ such that $\phi(a) = a'$, $\phi(b) = b'$

now $a' - b' = \phi(a) - \phi(b) = \phi(a-b) \in \phi(R)$ and

$a'b' = \phi(a)\phi(b) = \phi(ab) \in \phi(R)$

So, $\phi(R)$ is a subring of R'

Theorem 3.3.6 Let R and R' be two rings and $\phi: R \to R'$ be a onto homomorphism. If R be ~~a~~ a commutative ring, then R' is commutative.

Proof: Let $a', b' \in R'$. As ϕ is onto, $\exists\ a, b \in R$ such that $\phi(a) = a'$, $\phi(b) = b'$

now $a'.b' = \phi(a).\phi(b) = \phi(ab) = \phi(ba)$ (As R is commutative)

So, $a'.b' = \phi(b*a) = \phi(b).\phi(a) = b'.a'$

So R' is commutative

Corollary 3.3.7 ~~Note:~~ Let R and R' be two rings and $\phi : R \to R'$ be a homomorphism, then if R be commutative, then $\phi(R)$ is commutative.

Proof: Exercise.

Note. The converse of Theorem 3.3.6 is ~~d~~ not true. (Give example)

Theorem 3.3.8 Let R and R' be two rings and $\phi : R \to R'$ be a homomorphism. Then

(i) If S be a subring of R then $\phi(S)$ is a subring of R'

(ii) If S' be a subring of R' then $\phi^{-1}(S') = \{x \in R : \phi(x) \in S'\}$ is a subring of R.

Proof: $\phi(S) \neq \phi$ as $0' = \phi(0) \in \phi(S)$. Let $a', b' \in \phi(S)$

So, $\exists\, a, b \in S$ such that $\phi(a) = a'$, $\phi(b) = b'$

$a' - b' = \phi(a) - \phi(b) = \phi(a-b) \in \phi(S)$ as $a - b \in S$

$a'.b' = \phi(a).\phi(b) = \phi(ab) \in S$ as $ab \in S$

So, $\phi(S)$ is a subring of R'

(ii) Since S' is a subring of R', $0' \in S'$. Since $\phi(0) = 0'$ So, $0 \in \phi^{-1}(S')$ Hence $\phi^{-1}(S') \neq \phi$

Let $a, b \in \phi^{-1}(S')$. ~~So, $\exists\, x, y \in R$ such that~~ ~~$\phi(x) \in S'$~~

~~Now $a - b = \phi(x) - \phi(y) = \phi(x-y)$~~

So, $\phi(a) \in S'$ and $\phi(b) \in S'$. Since S' is a subring of R' $\phi(a), \phi(b) \in S' \Rightarrow \phi(a) - \phi(b) \in S$ and $\phi(a)\phi(b) \in S'$ $\Rightarrow \phi(a-b) \in S'$, $\phi(ab) \in S'$

$\Rightarrow a - b \in \phi^{-1}(S')$, $ab \in \phi^{-1}(S')$

So, $\phi^{-1}(S')$ is a subring of R.

Theorem 3.3.9 Let R and R' be two rings and $\phi : R \to R'$ be an onto homomorphism. Then

(i) if U be an ideal of R, then $\phi(U)$ is an ideal of R'

(ii) if U' be an ideal of R', then $\phi^{-1}(U') = \{x \in R : \phi(x) \in U'\}$ is an ideal of R.

Proof: (i) Since U is an ideal, U is a subring of R.

So, $\phi(U)$ is a subring of R'; by previous theorem.

Let $u' \in \phi(U)$ and $r' \in R'$. Then $\exists\, u \in U$, $r \in R$ such that $\phi(u) = u'$ and $\phi(r) = r'$ as ϕ is onto.

Since U is an ideal of R, $u \in U$, $r \in R \Rightarrow ru \in U$ and $ur \in U$

now $u'r' = \phi(u).\phi(r) = \phi(ur) \in \phi(U)$ as $ur \in U$

and $r'u' = \phi(r)\phi(u) = \phi(ru) \in \phi(U)$ as $ru \in U$

So, $\phi(U)$ is an ideal of R'

(ii) Exercise

Definition 3.3.10 Let R and R' be two rings and let $\phi : R \to R'$ be a homomorphism. The kernel of ϕ denoted by $\operatorname{Ker}\phi$ is a subset of R given by $\operatorname{Ker}\phi = \{a \in R : \phi(a) = 0'\}$, $0'$ is the zero element in R'

Theorem 3.3.11 Let R, and R' be two rings ~~are~~ and $\phi : R \to R'$ be a homomorphism. Then $\operatorname{Ker}\phi$ is an ideal of R.

As $\phi(0) = 0'$, 0 and $0'$ are the zero elements of R and R' respectively, $0 \in \operatorname{Ker}\phi$ and $\operatorname{Ker}\phi \neq \phi$. Let $a, b \in \operatorname{Ker}\phi$

then $\phi(a) = 0'$, $\phi(b) = 0'$ $\Rightarrow \phi(a-b) = \phi(a + (-b))$

$= \phi(a) - \phi(b) = 0'$

This shows that ~~a-b ∈ Ker φ~~ $a-b \in Ker\phi$

Let $r \in R$. Then $\phi(a \cdot r) = \phi(a) \cdot \phi(r) = 0' \cdot \phi(r) = 0'$

and $\phi(r \cdot a) = \phi(r) \cdot \phi(a) = \phi(r) \cdot 0' = 0'$

So, $a \in Ker\phi, r \in R \Rightarrow a \cdot r \in Ker\phi$ and $r \cdot a \in Ker\phi$

So, $Ker\phi$ is an ideal of R.

Theorem 3.3.12 Let R and R' be two rings and $\phi: R \to R'$ be an onto homomorphism. Then the ideals of R, each containing $Ker\phi$, are in ~~one-one~~ one-to-one correspondence with the ideals of R'.

Proof: Let $\mathcal{B}$ be the set of all ideals of R, each containing $Ker\phi$ and $\mathcal{B}'$ be the set of all ideals of R'.

Let $U \in \mathcal{B}$. Then U is an ideal of R and $Ker\phi \subseteq U$.

$\phi(U)$ is an ideal of R', i.e., $\phi(U) \in \mathcal{B}'$ ... (1)

Let $U' \in \mathcal{B}'$. Then U' is an ideal of R'. $\phi^{-1}(U')$ is an ideal of R. Let $a \in Ker\phi$. Then $\phi(a) = 0'$, the zero element of R' and so, $\phi(a) \in U'$. This implies $a \in \phi^{-1}(U')$. So, $Ker\phi \subseteq \phi^{-1}(U')$. Hence $\phi^{-1}(U') \in \mathcal{B}$ ... (2)

From (1) and (2), it follows that to each element of $\mathcal{B}$, there corresponds an element of $\mathcal{B}'$ and conversely.

Theorem 3.3.13 Let R and R' be two rings and $\phi: R \to R'$ be an onto homomorphism. Then ϕ is an isomorphism if and only if $Ker\phi = \{0\}$, 0 is the zero element of R.