Proof: Let ϕ be an isomorphism. Then ϕ is one-to-one. We have $\phi(0) = 0'$. Let $x \in \operatorname{Ker}\phi$. Then $\phi(x) = 0'$. So, $\phi(x) = \phi(0)$. So, $x = 0$. So, $\operatorname{Ker}\phi = \{0\}$.

Conversely, let $\operatorname{Ker}\phi = \{0\}$. Let $a, b \in R$ and $\phi(a) = \phi(b)$. Then $\phi(a-b) = \phi(a) - \phi(b) = 0'$. This implies $a-b \in \operatorname{Ker}\phi$. Since $\operatorname{Ker}\phi = \{0\}$, so $a-b = 0$ or $a = b$.

So, ϕ is one-to-one. As ϕ is an onto homomorphism, so, ϕ is an isomorphism.

Theorem 3.3.14 Let R and R' be two rings and $\phi : R \to R'$ be an isomorphism. Then

(i) R is commutative if and only R' is commutative

(ii) R is a ring with unity if and only R' is a ring with unity.

(iii) R is a ring without divisors of zero if and only if R' is a ring without divisors of zero.

(iv) $a \in R$ is a unit in R if and only if $\phi(a)$ is a unit in R'

Proof: (i) Let R be commutative. Since ϕ is an isomorphism, ϕ is an onto homomorphism and so, R' is commutative, by Theorem 3.3.6.

Conversely, let R' be commutative and let $a, b \in R$

$$\phi(a \cdot b) = \phi(a) \cdot \phi(b) = \phi(b) \cdot \phi(a) \text{ as } R' \text{ is commutative}$$
$$= \phi(b \cdot a), \text{ since } \phi \text{ is a homomorphism}$$

Since ϕ is an isomorphism, ϕ is one-to-one

So, $\phi(a \cdot b) = \phi(b \cdot a) \Rightarrow a \cdot b = b \cdot a$. So, R is commutative.

(ii) Let R be a ring with unity 1. Since ϕ is an isomorphism, it an onto homomorphism, so by Theorem 3.3.3, R' is a ring with unity $\phi(1)$.

Conversely, let R' be a ring with unity $1'$.

Let $\phi^{-1}(1') = 1$. Then $\phi(1) = 1'$

Let $a \in R$. Then $\phi(a.1) = \phi(a).\phi(1) = \phi(a).1' = \phi(a)$

and $\phi(1.a) = \phi(1).\phi(a) = 1'.\phi(a) = \phi(a)$

So, $\phi(a.1) = \phi(1.a) = \phi(a)$. As ϕ is an isomorphism, ϕ is one-to-one. So,

$$a.1 = 1.a = a \quad \text{for all } a \in R.$$

So, 1 is the unity in R.

(iii) Let R be a ring without divisors of zero. Let $a', b' \in R'$ and $a' \neq 0'$, $b' \neq 0'$. Then $\exists$ unique elements $a, b \in R$ such that $\phi(a) = a'$, $\phi(b) = b'$ as ϕ is an isomorphism. As ϕ is an isomorphism, $a \neq 0$, $b \neq 0$ (0, $0'$ are the zero elements of R and R' respectively), because,

$a = 0 \Rightarrow \phi(a) = \phi(0) = 0' \Rightarrow a' = 0$, a contradiction

and $b = 0 \Rightarrow \phi(b) = \phi(0) = 0' \Rightarrow b' = 0$, a contradiction.

Since R contains no divisor of zero, $a \neq 0, b \neq 0 \Rightarrow ab \neq 0$

$\Rightarrow \phi(ab) \neq \phi(0) = 0'$. So, $\phi(a).\phi(b) \neq 0'$. So, $a'.b' \neq 0'$

So, R' contains no divisor of zero.

Conversely, let R' contains no divisor of zero.

Let $a, b \in R$ and $a \neq 0$, $b \neq 0$. As ϕ is an isomorphism, $\phi(a) \neq \phi(0) = 0'$ and $\phi(b) \neq \phi(0) = 0'$.

As $\phi(a) \neq 0'$ and $\phi(b) \neq 0'$, so, $\phi(a)\cdot\phi(b) \neq 0'$ as R' has no divisor of zero. So, $\phi(ab) \neq 0'$

Then $ab \neq 0$, because if $ab = 0$, $\phi(ab) = \phi(0) = 0'$, as ϕ is an isomorphism, a contradiction.

So, $ab \neq 0$. Hence R has no divisor of zero.

(iv) Let a be a unit in R. Since ϕ is an onto homomorphism, it follows from Theorem 3.3.3, $\phi(a)$ is a unit in R'

Conversely, let $a \in R$ and $\phi(a)$ is a unit in R'

So, $\exists\ b' \in R'$ such that $\phi(a)\cdot b' = b'\cdot\phi(a) = 1'$, $1'$ is the unity in R'.

As ϕ is an isomorphism, $\exists$ an element $b \in R$ such that $\phi(b) = b'$

So, $\phi(a)\cdot\phi(b) = \phi(b)\cdot\phi(a) = 1'$

So, $\phi(ab) = \phi(ba) = \phi(1) = 1'$ as $\phi(1) = 1'$

As ϕ is one-to-one $ab = ba = 1$, 1 the unity in R, (1 exists in R as R is a ring with unity (by (ii)))

So, $\exists\ b \in R$ such that $ab = ba = 1$

So, a is unit in R.

Theorem 3.3.15. Let U be an ideal of a ring R. Then the mapping $\phi : R \to R/U$ defined by $\phi(x) = x + U$ is an onto homomorphism with Kernel U.

Proof: Let $x, y \in R$. Then $\phi(x) = x + U$, $\phi(y) = y + U$

$\phi(x+y) = (x+y) + U = (x+U) + (y+U) = \phi(x) + \phi(y)$

$\phi(xy) = xy + U = (x+U)(y+U) = \phi(x)\phi(y)$

This proves that ϕ is a homomorphism.

The zero element in the quotient ring R/U is U.

$\text{Ker}\,\phi = \{x \in R : \phi(x) = U\}$. Now $\phi(x) = U \Leftrightarrow x + U = U$

$\Leftrightarrow x \in U$. So, $\text{Ker}\,\phi = U$.

Theorem 3.3.16 (First isomorphism theorem)

Let R and S be two rings and $f : R \to S$ be an onto homomorphism. Then $S \simeq R/_{\text{Ker} f}$

Proof: Let $I = \text{Ker} f$. Define a mapping $\psi : R/_I \to S$ by $\psi(a+I) = f(a)$

We need to verify that ψ is well defined. Let $a, b \in R$ be such that $a + I = b + I$. Then $a - b \in I = \text{Ker} f$

So, $f(a-b) = 0'$, $0'$ is the zero element in S

But this implies $f(a) - f(b) = 0'$ or, $f(a) = f(b)$

or, $\psi(a+I) = \psi(b+I)$. So, ψ is well defined.

Now let $a, b \in R$ and $\psi((a+I)+(b+I)) = \psi((a+b)+I)$

$= f(a+b) = f(a) + f(b) = \psi(a+I) + \psi(b+I)$

Also $\psi((a+I)(b+I)) = \psi(ab+I) = f(ab) = f(a)f(b)$

$= \psi(a+I)\psi(b+I)$. So, ψ is a homomorphism

Next consider any $y \in S$. As f is onto, we have, $\exists\, x \in R$ such that $f(x) = y$ Then $y = \psi(x+I)$.

Finally, $\text{Ker}\,\psi = \{a + I \in R/_I : \psi(a+I) = 0'\}$

$= \{a + I \in R/_I : f(a) = 0'\} = \{a + I \in R/_I : a \in I\}$ (since $I = \text{Ker} f$)

$= \{I\}$. So, ψ is injective. Hence ψ is an isomorphism. Hence $S \simeq R/_{\text{Ker} f}$