Theorem 2.1.3 A nonempty set W of a vector space V over a field F is a subspace if and only if

(i) $\alpha, \beta \in W \Rightarrow \alpha + \beta \in W$

(ii) $c \in F, \alpha \in W \Rightarrow c\alpha \in W$

Proof: Let the conditions hold in W.

Let $\alpha, \beta \in W$. Since F is a field $-1 \in F$, where 1 is the identity element in F. By (ii) $(-1)\beta \in W \Rightarrow -\beta \in W$

Then by (i) $\alpha + (-\beta) \in W$ or, $\alpha - \beta \in W$

Thus $\alpha, \beta \in W \Rightarrow \alpha - \beta \in W$

This proves that W is a subgroup of the additive group V, Since V is a commutative group, W is also a commutative subgroup of V.

So, conditions V1 to V5 for a vector space are satisfied in W. V6 is satisfied in W by (ii). The conditions V7 to V10 are satisfied in W, since they are hereditary properties. So, W is itself a vector space over F and so W is a subspace of V.

The necessity of the conditions (i) and (ii) follows from the definition of a vector space.

Theorem 2.1.4 Let $\{W_i : i \in I\}$ be a collection of subspaces of a vector space V over the field F. Then

$$\bigcap_{i \in I} W_i \text{ is also a subspace of } V.$$

Proof ~~As~~ As $\theta \in W_i$ for each $i \in I$. So, $\theta \in \bigcap_{i \in I} W_i$

So, $\bigcap_{i \in I} W_i$ is non-empty. Let $\alpha, \beta \in \bigcap_{i \in I} W_i$

So, $\alpha, \beta \in W_i$ for each $i \in I$. As W_i is a subspace

So, $\alpha+\beta \in W_i$ and $c\alpha \in W_i$ for any $c \in F$ for each i.

So, $c\alpha \in \bigcap_{i \in I} W_i$ and $\alpha+\beta \in \bigcap_{i\in I} W_i$

So, $\bigcap_{i \in I} W_i$ is a subspace by Theorem 2-1.3

Two equivalent Theorems which is equivalent to Theorem 2.1.3., whose proofs are similar to Theorem 2.1.3, are

Theorem 2.1.5 A non-empty subset W of a vector space V over a field F is a subspace if and only if

$\alpha, \beta \in W$ and $c, d \in F \Rightarrow c\alpha + d\beta \in W$

Theorem 2.1.6 A non-empty subset W of a vector space V over a field F is a subspace if and only if $\alpha, \beta \in W$ and $c \in F \Rightarrow c\alpha + \beta \in W$.

Linear sum of two subspaces

Let U and W be two subspaces of a vector space V over a field F. Then the subset $\{u+w : u \in U, w \in W\}$ is said to be the linear ~~subspace~~ sum of the subspaces and denoted by $U+W$.

Let $u \in U$. Then $u = u + \theta$, where $u \in U$ and $\theta \in W$. So, $u \in U+W$. So, $U \subset U+W$. Similarly $W \subset U+W$.

Theorem 2.1.7 Let U and W be two subspaces of a vector space V over a field F. Then the linear sum $U+W$ is a subspace of V and it is the smallest subspace of V containing U and W

Proof: As $\theta \in U$, $\theta \in W \Rightarrow \theta + \theta = \theta \in U + W$.

So $U+W$ is non-empty. Let $\alpha_1, \alpha_2 \in U+W$

So, $\alpha_1 = u_1 + w_1$, $\alpha_2 = u_2 + w_2$, $u_1, u_2 \in U$ and $w_1, w_2 \in W$

So, $\alpha_1 + \alpha_2 = (u_1 + w_1) + (u_2 + w_2) = (u_1 + u_2) + (w_1 + w_2) \in U + W$

as $u_1 + u_2 \in U$ and $w_1 + w_2 \in W$ as they are subspaces.

Let $c \in F$ and $\alpha \in U + W$. Let $\alpha = u + w$, $u \in U$, $w \in W$

$c\alpha = c(u+w) = cu + cw \in U + W$ as $cu \in U$ and $cw \in W$

(U, W are subspaces)

So, $U+W$ is a subspace of V.

Let P be any subspace of V containing U and W

Let $\alpha \in U+W$. Then $\alpha = u + w$, $u \in U$, $w \in W$

Since $U \subset P$, $u \in P$ and since $W \subset P$, $w \in P$

Since P is a subspace, $u + w \in P \Rightarrow \alpha \in P$

This proves that $U + W \subset P$

As $U + W$ contains U and V. So, $U + W$ is the smallest subspace containing U and W.

Theorem 2.1.8. Let V be a vector space over a field F. Let $S = \{\alpha_1, \alpha_2, \dots, \alpha_r\} \subset V$

Then the set $W = \{ \sum_{i=1}^{r} c_i \alpha_i : c_i \in F, i = 1, 2, \dots r \}$ is a subspace of V. Let $c_1 = 1$, $c_2 = c_3 = \dots = c_r = 0$ then

$\sum c_i \alpha_i = \alpha_1 \in W$. So, W is non-empty.

Let $\alpha, \beta \in W$ Then $\alpha = \sum_{i=1}^{r} a_i \alpha_i$, $a_i \in F$, $i = 1, 2, \dots, r$

and $\beta = \sum_{i=1}^{r} b_i \alpha_i$, $b_i \in F$, $i = 1, 2, \dots, r$.

$\alpha+\beta = \sum_{i=1}^{r} a_i\alpha_i + \sum_{i=1}^{r} b_i\alpha_i = \sum_{i=1}^{r}(a_i+b_i)\alpha_i = \sum c_i\alpha_i \in W$

where

$c_i = a_i+b_i \in F$, $i=1,2,\ldots,r$

Let $c\in F$. Now $c\alpha = c\sum_{i=1}^{r} a_i\alpha_i = \sum_{i=1}^{r}(ca_i)\alpha_i = \sum_{i=1}^{r} d_i\alpha_i \in W$

where $d_i = ca_i \in F$, $i=1,2,\ldots,r$

So, W is a subspace of V.

Note1: It is the smallest subspace containing $S=\{\alpha_1,\alpha_2,\ldots,\alpha_r\}$

Proof: Let P be any subspace of V containing S

Let $\alpha\in W$. Then $\alpha = \sum_{i=1}^{r} c_i\alpha_i$, $c_i\in F$, $i=1,2,\ldots,r$

Since P is a subspace containing S and it contains α_i, $i=1,2,\ldots,r$

So, $c_1\alpha_1, c_2\alpha_2, \ldots, c_n\alpha_n \in P$. So $\sum_{i=1}^{r} c_i\alpha_i \in P$

So, $W\subset P$. So, W is the smallest subspace containing $S=\{\alpha_1,\alpha_2,\ldots,\alpha_r\}$

Note2: The smallest subspace containing S is the intersection of all subspaces containing S.

Definition 2.1.9 The smallest subspace containing a finite set S of a vector space V is said to be the linear span of S and and is denoted by L(S) or Span(S). L(S) is said to be generated (or spanned) by the set S and S is said to be the generating set of L(S)

Definition 2.1.10 Let V be a vector space over the field F. Let S be a non-empty subset of V. A vector $v\in V$ is called a linear combination of vectors of S if $\exists$ a finite number of vectors $u_1,u_2,\ldots,u_n$ in S and scalars $a_1,a_2,\ldots,a_n$ in F such that $v = a_1u_1+a_2u_2+\cdots+a_nu_n$. In this case we also say that v is a linear combination of $u_1,u_2,\ldots,u_n$ and call $a_1,a_2,\ldots,a_n$ as the coefficients of the linear combinations.