Note 1. For convenience, we take $span(\phi) = \{\theta\}$, ϕ is the null set.

Note 2. For any subset S of a vector space V over F, we define $Span(S)$ as $Span(S) = \{ \sum_{i=1}^{r} c_i \alpha_i : c_i \in F, \alpha_i \in V, i = 1, 2, \ldots, r$ and r is a positive integer $\}$

We can easily check that it is a subspace of V.

Definition 2.1.10 A subset S of a vector space V over a field F generates (or spans) V if $span(S) = V$. In this case, we also say that the vectors of S generate (or span) V.

Example 1. The vectors $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$ generates $\mathbb{R}^3$ since an arbitrary vector (a_1, a_2, a_3) in $\mathbb{R}^3$ is a linear combination of the three given vectors. In fact the scalars r, s, and t for which

$$r(1,1,0) + s(1,0,1) + t(0,1,1) = (a_1, a_2, a_3)$$

are $r = \frac{1}{2}(a_1 + a_2 - a_3)$, $s = \frac{1}{2}(a_1 - a_2 + a_3)$ and $t = \frac{1}{2}(-a_1 + a_2 + a_3)$

Example 2 The polynomials $x^2 + 3x - 2$, $2x^2 + 5x - 3$ and $-x^2 - 4x + 4$ generate $P_2(\mathbb{R})$ (vector space of all real polynomials of degree at most 2) and each polynomial $ax^2 + bx + c$ in $P_2(\mathbb{R})$ is a linear combination of these three. In fact,

$$(-8a + 5b + 3c)(x^2 + 3x - 2) + (4a - 2b - c)(2x^2 + 5x - 3) + (-a + b + c)(-x^2 - 4x + 4) = ax^2 + bx + c$$

Definition 2.1.11 (Linear dependence and independence)

A finite set of vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of a vector space V over a field F is said to be linearly dependent in V if there exists scalars $c_1, c_2, \ldots, c_n$ not all zero, in F such that

$$\sum_{i=1}^{n} c_i \alpha_i = \theta \quad \ldots \text{(i)}$$

Otherwise, $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is said to be linearly independent.

So, $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is said to be linearly independent if the equality (i) is satisfied only when $c_1 = c_2 = \cdots = c_n = 0$

An arbitrary set S of vectors of a vector space V over a field F is said to be linearly dependent in V if there exists a finite subset of S which is linearly dependent in V.

A set of vectors which is not linearly dependent is said to be linearly ~~dependent~~ independent.

Theorem 2.1.12 A superset of a linearly dependent set of vectors is linearly dependent and a subset of a linearly independent set is linearly independent.

Note: The set ϕ is taken as linearly independent.

Example 1 Any set containing the null vector θ is linearly dependent in any vector space V as $1 \cdot \theta = \theta$

Example 2 The set $\{(1,2,3), (0,3,5), (0,0,7)\}$ is linearly independent (~~check it~~) in $\mathbb{R}^3$ (check it)

Example 3 The set $\{(1,1,3), (2,1,4), (-1,0,-1)\}$ is linearly dependent (check it)

Worked Examples

1. Examine if the set of vectors $\{(2,1,1), (1,2,2), (1,1,1)\}$ is linearly dependent in $\mathbb{R}^3$.

Solution: Here $(2,1,1) + (1,2,2) = (3,3,3) = 3(1,1,1)$

$$\text{So,} \quad (2,1,1) + (1,2,2) - 3(1,1,1) = (0,0,0) = \theta$$

So the set is linearly dependent.

2. Prove that the set ~~of~~ of vectors $\{(1,2,2),(2,1,2),(2,2,1)\}$ is linearly independent in $\mathbb{R}^3$

Solution: Let $a(1,2,2)+b(2,1,2)+c(2,2,1)=(0,0,0)$, $a,b,c \in \mathbb{R}$

~~for some~~

Then $a+2b+2c=0$, $2a+b+2c=0$, $2a+2b+c=0$

This is a homogeneous system of three equation in a, b, c

Here coefficient determinant $\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 5 \neq 0$

So, the system has unique solution, $a=0$, $b=0$, $c=0$

This proves that the set $\{(1,2,2),(2,1,2),(2,2,1)\}$ is linearly independent.

Note: The union of two subspaces of V is not, in general, a subspace of V. For example, let us consider ~~the and~~ the two subspaces S and T ~~of~~ of the vector space $\mathbb{R}^3$, where $S=\{(x,y,z)\in\mathbb{R}^3 : y=0, z=0\}$ and $T=\{(x,y,z)\in\mathbb{R}^3 : x=0, z=0\}$

Here ~~also~~ $\alpha=(1,0,0)\in S$ and $\beta=(0,1,0)\in T$

Then $\alpha+\beta=(1,1,0)\notin S\cup T$. So, $S\cup T$ is not a subspace of $\mathbb{R}^3$.

Theorem 2.1.13 If U and W be two subspaces of a vector space V over a field F, then the union $U\cup W$ is a subspace of V if and only if either $U\subset W$ or $W\subset U$.

Proof: Let $U\cup W$ be a subspace of V. We prove either $U\subset W$ or, $W\subset U$, i.e., either $U-W=\phi$ or $W-U=\phi$

Let us assume that both $U-W \neq \phi$ and $W-U \neq \phi$. Then $\exists$ a vector α such that $\alpha \in U$, but $\alpha \notin W$ and a vector $\beta \in W$ but $\beta \notin U$

$\alpha \in U \Rightarrow \alpha \in U \cup W$ and $\beta \in W \Rightarrow \beta \in U \cup W$

Since $U \cup W$ is a subspace of V, $\alpha+\beta \in U \cup W$.

This implies $\alpha+\beta \in U$ or, $\alpha+\beta \in W$.

$\alpha+\beta \in U$ and $\alpha \in U \Rightarrow (\alpha+\beta)-\alpha \in U$, since U is a subspace $\Rightarrow \beta \in U$, a contradiction

$\alpha+\beta \in W$ and $\beta \in W \Rightarrow (\alpha+\beta)-\beta \in W$, since W is a subspace $\Rightarrow \alpha \in W$, a contradiction.

So, $\alpha+\beta \notin U$, $\alpha+\beta \notin W$ and so, $\alpha+\beta \notin U \cup W$.

So, our assumption that both $U-W \neq 0$ and $W-U \neq \phi$ is not true. So, either $U-W=\phi$ or $W-U=\phi$.

So, either $U \subset W$ or $W \subset U$.

Theorem If S and T be two non-empty finite subsets of a vector space V over a field F and $S \subset T$, then $L(S) \subset L(T)$.

Proof: Let $S=\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and let $x \in L(S)$

Then $x = \sum_{i=1}^{n} a_i \alpha_i$ for $a_i \in F$, $i=1,2,\ldots,n$

As $\alpha_i \in S$, $\alpha_i \in T$ as $S \subset T$, for $i=1,2,\ldots,n$

So, $a_i\alpha_i \in L(T)$ as $a_i \in F$, $i=1,\ldots n$ and $L(T)$ is a subspace of V. So, $\sum_{i=1}^{n} a_i\alpha_i \in L(T)$ as $L(T)$ is a subspace of V. So, $x \in L(T)$.

So, $L(S) \subset L(T)$.