Corollary 2.1.14 If S and T be any two non-empty subsets of a vector space V over a field F and $S \subset T$, then $L(S) \subset L(T)$

Proof: Proof is similar as before.

Theorem 2.1.15 If S and T be two non-empty finite subsets of a vector space over a field F and each element of T is a linear combination of the vectors of S, then $L(T) \subset L(S)$.

Proof: Let $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, $T = \{\beta_1, \beta_2, \dots \beta_m\}$

and let $\beta_i = c_{i1}\alpha_1 + c_{i2}\alpha_2 + \dots + c_{in}\alpha_n = \sum_{r=1}^{n} c_{ir}\alpha_r$ for $c_{ij} \in F$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$

Let $x \in L(T)$. So, $x = \sum_{i=1}^{m} p_i\beta_i$, $p_i \in F$, $i = 1, 2, \dots, m$

$$\text{So, } x = \sum_{i=1}^{m} p_i\left(\sum_{r=1}^{n} c_{ir}\alpha_r\right)$$

$$= p_1(c_{11}\alpha_1 + c_{12}\alpha_2 + \dots + c_{1n}\alpha_n) + p_2(c_{21}\alpha_1 + c_{22}\alpha_2 + \dots + c_{2n}\alpha_n) + \dots + p_m(c_{m1}\alpha_1 + c_{m2}\alpha_2 + \dots + c_{mn}\alpha_n)$$

$$= \sum_{j=1}^{n} d_j\alpha_j \quad \text{where } d_j = \sum_{i=1}^{m} p_i c_{ij}, \; j = 1, 2, \dots, n$$

So, $x \in L(S)$. So, $L(T) \subset L(S)$

Note: The theorem says that if $T \subset L(S)$ then $L(T) \subset L(S)$

Worked out Examples

1. In $\mathbb{R}^3$, $\alpha = (4, 3, 5)$, $\beta = (0, 1, 3)$ and $\gamma = (2, 1, 1)$

Examine if α is a linear combination of β and γ

Solution: Let $\alpha = c\beta + d\gamma$ where $c, d \in \mathbb{R}$

Then $(4, 3, 5) = c(0, 1, 3) + d(2, 1, 1) = (2d, c+d, 3c+d)$

So, $2d = 4$, $c+d = 3$, $3c+d = 5$ giving $c = 1, d = 2$

Hence $\alpha = \beta + 2\gamma$ and α is a linear combination of β and γ.

2. Determine the subspace of $\mathbb{R}^3$ spanned by the vectors $\alpha = (1,2,3)$, $\beta = (3,1,0)$. Examine if $\gamma = (-1,3,6)$ is in the subspace.

Solution: $L(\{\alpha,\beta\}) = \{c\alpha + d\beta : c, d \in \mathbb{R}\}$

If $\gamma \in L(\{\alpha,\beta\})$ then $\exists\ c, d \in \mathbb{R}$

such that $\gamma = (-1,3,6) = c\alpha + d\beta = (c+3d,\ 2c+d,\ 3c)$

So, $c+3d = -1$, $2c+d = 3$ and $3c = 6$

So, $c = 2$ from the third equation. So $d = -1$, from the first equation and $d = -1$ from the third equation. So, $c = 2, d = -1$

Hence $\gamma = 2\alpha - \beta \in L(\{\alpha,\beta\})$

Theorem 2.1.16 If S be a non-empty subset of a vector space V over a field F then $L(L(S)) = L(S)$

Proof: Exercise

Theorem 2.1.17 Let S and T be two non-empty subsets of a vector space V over a field F. Then

$$L(S \cup T) = L(S) + L(T)$$

Let $x \in L(S) + L(T)$. Then $x = y + z$ where $y \in L(S)$ and $z \in L(T)$

So $y = \sum_{i=1}^{m} c_i\alpha_i$, $z = \sum_{i=1}^{n} d_j\beta_j$ where $\alpha_i \in S$, $i = 1,2,\ldots,m$

and $\beta_j \in S$, $j = 1,2,\ldots,n$ $c_i, d_j \in F$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$

As $\alpha_i, \beta_j \in S \cup T$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$

So $x = \sum_{i=1}^{m} c_i\alpha_i + \sum_{j=1}^{n} d_j\beta_j \in L(S \cup T)$

So, $L(S)+L(T) \subset L(S \cup T)$

Now, $S \subset L(S) \Rightarrow S \subset L(S)+L(T)$

and $T \subset L(T) \Rightarrow T \subset L(S)+L(T)$

So, $S \cup T \subset L(S)+L(T)$

Hence $L(S \cup T) = L(S)+L(T)$

Note: In particular, if S and T are subspaces of V then $L(S)=S$ and $L(T)=T$. So, if S and T are subspaces of V then $L(S \cup T) = S+T$ by the theorem.

<u>Worked out Examples</u>

1. Examine if the set S is a subspace of $\mathbb{R}^3$

(i) $S = \{(x,y,z) \in \mathbb{R}^3 : x=0\}$

(ii) $S = \{(x,y,z) \in \mathbb{R}^3 : xy=z\}$

Solution: (i) $(0,0,0) \in S$. So, S is non-empty. Let $\alpha, \beta \in S$. Then $\alpha = (0,a,b)$ and $\beta = (0,c,d)$ $a,b,c,d \in \mathbb{R}$

So, Then $\alpha+\beta = (0, a+c, b+d) \in S$

Now $c\alpha = c(0,a,b) = (0, ac, bc) \in S$

So, S is a subspace of $\mathbb{R}^3$

(ii) $(1,1,1) \in S$ as $1 \times 1 = 1$, $(1,2,2) \in S$ as $1 \times 2 = 2$

Now $(1,1,1)+(1,2,2) = (2,3,3)$ but $2 \times 3 = 6 \neq 3$

So, $(2,3,3) \notin S$. So, S is not a subspace of $\mathbb{R}^3$

2. Examine if the set S is a subspace of the vector space $M_{2\times2}(\mathbb{R})$, where

(i) S is the set of all 2×2 symmetric matrix

(ii) S is the set of all 2×2 skew-symmetric matrix

Solution : (i) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S$ as $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is symmetric. So, S is non-empty.

Let $A, B \in S$. Then $A^t = A$ and $B^t = B$

So. $(A+B)^t = A^t + B^t = A + B$. So, $A+B$ is symmetric. So, $A+B \in S$. Let $c \in \mathbb{R}$ and $A \in S$

Then So. $A^t = A$. Now $(cA)^t = cA^t = cA$

So, cA is symmetric. So, $cA \in S$

Hence S is a subspace of $M_{2\times 2}(\mathbb{R})$

(ii) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S$ as $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is skew-symmetric also.

So, S is non-empty. Let $A, B \in S$. Then

$A^t = -A$, $B^t = -B$. So, $(A+B)^t = A^t + B^t = -A - B$
$= -(A+B)$

So, $A+B$ is skew-symmetric. So, $A+B \in S$

Let $c \in \mathbb{R}$ and $A \in S$. Then $A^t = -A$.

So, $(cA)^t = cA^t = c(-A) = -cA$ So, cA is skew-symmetric. Hence $cA \in S$. So, S is a subspace of $M_{2\times 2}(\mathbb{R})$.

3. Show that the set S is a subspace of the vector space $C[0,1]$ (the vector space of all real valued continuous function on $[0,1]$ over $\mathbb{R}$) where $S = \{f \in C[0,1] : f(0) = 0\}$

Solution As Zero function $0 \in S$ so, S is non-empty. Let $f, g \in S$ then $f(0) = 0$ $g(0) = 0$. So, $(f+g)(0) = f(0) + g(0) = 0 + 0 = 0$

So, Also, as f, g are continuous, $f+g$ is continuous.

So, $f+g \in S$. Let $c \in \mathbb{R}$ and $f \in S$. So, $f(0) = 0$

So, cf is continuous and $(cf)(0) = c(f(0)) = c\cdot 0 = 0$

So, $cf \in S$. So, S is a subspace of $C[0,1]$.