Theorem 2.1.17 If the set of vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ in a vector space over a field F be linearly dependent, then at least one of the vectors of the set can be expressed as a linear combination of the remaining others.
Conversely, if one of the vectors of the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a linear combination of the remaining others, the set is linearly dependent.

Proof: ~~Since~~ Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be linearly dependent. So, $\exists$ scalars $c_1, c_2, \ldots, c_n$ in F, not all zero, such that

$$\sum_{i=1}^{n} c_i \alpha_i = \theta$$

Let $c_j \neq 0$. Then $c_j^{-1} \in F$ and $c_j^{-1} c_j = 1$, 1 being the identity element in F.

Now $c_j \alpha_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} c_i \alpha_i$ or, $\alpha_j = c_j^{-1} \left(\sum_{\substack{i=1 \\ i \neq j}}^{n} c_i \alpha_i \right)$

$$= \sum_{\substack{i=1 \\ i \neq j}}^{n} (c_j^{-1} c_i) \alpha_i = \sum_{\substack{i=1 \\ i \neq j}}^{n} d_i \alpha_i \quad \text{where } d_i = c_j^{-1} c_i, \ i = 1, 2, \ldots j-1, j+1, \ldots n$$

This shows that α_j is a linear combination of the remaining vectors $\alpha_1, \alpha_2, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_n$

Conversely, let one of the vectors, say α_j, is a linear combination of the other vectors of the set.

So, $\alpha_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} a_i \alpha_i$

or, $a_1 \alpha_1 + a_2 \alpha_2 + \cdots + a_{j-1} \alpha_{j-1} + (-1) \alpha_j + a_{j+1} \alpha_{j+1} + \cdots + a_n \alpha_n = \theta$

So, the above equality holds for scalars $a_1, a_2, \ldots a_{j-1}, -1, a_j, \ldots, a_n$ in F and since one of them is at least non-zero, the set is linearly dependent.

Theorem 2.1.18 A set of vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ where $\alpha_i \neq \theta$, $i=1,2,\ldots,n$, in a vector space V over a field F is linearly dependent if and only if $\exists$ a vector in the set which is a linear combination of the preceding others.

Proof: Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be linearly dependent. So, $\exists$ scalars $c_1, c_2, \ldots, c_n$, not all zero, in F such that

$$\sum c_i \alpha_i = \theta$$

Let i be the largest subscript such that $c_i \neq 0$

Then $i \neq 1$ as $\alpha_1 \neq \theta$. So, we have

$$c_i \alpha_i = \sum_{j=1}^{i-1} (-c_j)\alpha_j$$. Since $c_i \neq 0$, $c_i^{-1} \in F$ and so,

$$\alpha_i = \sum_{j=1}^{i-1} (-c_i^{-1} c_j)\alpha_j$$. So, α_i is a linear combination of the preceding vectors of the set.

Conversely, let α_j be the linear combination of the preceding vectors of the set. So,

$$\alpha_j = \sum_{k=1}^{j-1} d_k \alpha_k \quad \text{for scalars } d_k \in F,\ k=1,\ldots j-1$$

So, $d_1\alpha_1 + d_2\alpha_2 + \cdots + d_{j-1}\alpha_{j-1} - \alpha_j + 0.\alpha_{j+1} + 0.\alpha_{j+2} + \cdots + 0.\alpha_n = \theta$

Since the above equality holds for scalars $d_1, d_2, \ldots d_{j-1}, -1$, and 0 in F and ~~one of them~~ at least one of them is non-zero, the set of vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is linearly dependent.

Theorem 2.1.19 (Deletion Theorem)

If a non-null vector space V over a field F be spanned by a linearly dependent set $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, then V can also be spanned by a suitable proper subset of S.

Proof: Since S is linearly dependent, at least one of the vectors of the set, say α_j, can be expressed as a linear combination of the remaining others.

Let $\alpha_j = \sum_{\substack{i=1 \\ i\neq j}}^{n} d_i\alpha_i$ for some scalars $d_i \in F$, $i=1,2,\ldots j-1, j+1, \ldots, n$ —(i)

Here $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $L(S) = V$.

Let $T = \{\alpha_1, \alpha_2, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, n\}$

As T is a subset of S, $L(T) \subset L(S)$ by Theorem 2.1.13.

Using (i), we see that each element of S is a linear combination of elements of T. So, $L(S) \subset L(T)$ by Theorem 2.1.15

Hence $L(T) = L(S)$

Thus $V = L(T)$ and V is spanned by a proper subset of S.

Theorem 2.1.20. If $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a linearly independent set of generators of a vector space V, then no proper subset of S can be a spanning set of V.

Proof: Exercise.

Worked Example :1. Let $\{ \alpha_1 = (1,2,0), \alpha_2 = (3,-1,1), \alpha_3 = (4,1,1)$. Show that $S = \{\alpha_1, \alpha_2, \alpha_3\}$ is linearly dependent. Apply Deletion theorem to find a proper subset of S that generates $L(S)$

Solution: $c_1\alpha_1 + c_2\alpha_3 + c_3\alpha_3 = \theta$ where $c_1, c_2, c_3 \in \mathbb{R}$

Then $c_1(1,2,0) + c_2(3,-1,1) + c_3(4,1,1) = (0,0,0)$

So, $c_1 + 3c_2 + 4c_3 = 0$, $2c_1 - c_2 + c_3 = 0$, $c_2 + c_3 = 0$

This gives, $c_1 = c_2 = -c_3$. Taking $c_1 = 1$, $c_2 = 1$, $c_3 = -1$,

we have $\alpha_1 + \alpha_2 - \alpha_3 = (0,0,0)$

This shows that S is linearly dependent.

Now $\alpha_3 = \alpha_1 + \alpha_2$. By deletion theorem, α_3 can be deleted from the generating set of $L(S)$.

So, $L(\{\alpha_1, \alpha_2\}) = L(S)$

2.2 Basis and dimension of a vector space.

Let V be a vector space over a field F. V is said to be finitely generated or finite dimensional if $\exists$ a finite set of vectors in V generating V. Otherwise, V is said to be infinite dimensional. We shall be mainly concerned with finite dimensional ~~vs~~ space here.

Definition 2.2.1 (Basis) Let V be a vector space over a field F. A set S of vectors in V is said to be a basis of V if

(i) S is linearly independent and

(ii) $L(S) = V$

Theorem 2.2.1 There exists a basis for every finitely generated vector space

Proof: Case 1 Let V be a finitely generated vector space other than null space.

Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a finite set of generators of V. If S is linearly independent, then S