itself is a basis of V and the theorem is done.

If S is linearly dependent, then we can delete, by Deletion Theorem, at least one vector from S and obtain a proper subset S_1 of S, spanning the vector space V. If S_1 is linearly independent, then S_1 is a basis of V and the theorem is proved.

If S_1 be not linearly independent then we repeat the process of deletion and finally obtain, after $k(<n)$ steps of deletion, a subset S_k which is linearly independent in V and also spans V.

This is possible, because S is a finite set of n elements and in the extreme unfavourable case we can come, after n-1 steps of deletion, to a subset S_{n-1} containing only one non-zero vector (as V is non-null) that generates V and that is linearly independent.

Therefore our assertion that S_k is linearly independent set for some $k(<n)$ is true and hence it is a basis of V.

Case 2 Let $V=\{\theta\}$. Since ϕ is linearly independent and $L(\phi)=\theta$, ϕ is a basis of V.

This completes the proof.

Examples 1. $\{(1,0),(0,1)\}$ is a basis of $\mathbb{R}^2$

As $c_1(1,0)+c_2(0,1)=(0,0)$ implies $c_1=c_2=0$, so $S=\{(1,0),(0,1)\}$ is linearly independent. Let $(x,y)\in\mathbb{R}^2$. Then $(x,y)=x(1,0)+y(0,1)$. So $L(S)=\mathbb{R}^2$. So S is a basis of $\mathbb{R}^2$

Example 2 In F^n, let $e_1 = (1,0,0,\ldots,0)$, $e_2 = (0,1,0,\ldots,0)$, $\ldots$, $e_n = (0,0,\ldots,1)$. Then $\{e_1, e_2, \ldots, e_n\}$ is readily seen to be a basis of F^n (Prove it). It is called the standard basis of F^n.

Example 3. In $M_{m\times n}(F)$, let E_{ij} denote the matrix whose only non-zero entry is 1 in the ith row and jth column. Then the set $\{E_{ij} : i=1,2,\ldots,m,\ j=1,\ldots,n\}$ is a basis of $M_{m\times n}(F)$ (Prove it)

Example 4 In $P_n(F)$, the set $\{1, x, x^2, \ldots x^n\}$ is a basis. We call this basis, the standard basis for $P_n(F)$

Theorem 2.2.2 (Replacement theorem). If $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of a vector space V over a field F and a non-zero vector β of V is expressed as $\beta = \sum_{i=1}^{n} c_i\alpha_i$, $c_i \in F$, $i=1,2,\ldots,n$, then if $c_j \neq 0$, $\{\alpha_1, \alpha_2, \ldots, \alpha_{j-1}, \beta, \alpha_{j+1}, \ldots, \alpha_n\}$ is a new basis of V. (That is β can replaces α_j in the basis).

Proof: $\beta = \sum_{i=1}^{n} c_i\alpha_i$. So, $c_j\alpha_j = \sum_{\substack{i=1\\ i\neq j}}^{n} c_i\alpha_i + c_j^{-1}\beta$. As $c_j \neq 0$, $c_j^{-1} \in F$ and $c_jc_j^{-1} = 1$, 1 being the identity in F.

$$\text{So,}\quad \alpha_j = \sum_{\substack{i=1\\ i\neq j}}^{n} (c_j^{-1}c_i)\alpha_i + c_j^{-1}\beta = \sum_{\substack{i=1\\ i\neq j}}^{n} p_i\alpha_i + c_j^{-1}\beta \quad\text{—— (1)}$$

where $p_i = c_j^{-1}c_i$, $i=1,2,\ldots,j-1,j+1,\ldots,n$ and $p_j = c_j^{-1}$

We ~~first~~ first show that $\{\alpha_1, \alpha_2, \ldots, \alpha_{j-1}, \beta, \alpha_{j+1}, \ldots, \alpha_n\}$ is linearly independent. Let $\sum_{\substack{i=1 \\ i\neq j}}^{n} d_i\alpha_i + d_j\beta = \theta$

Then, $\sum_{\substack{i=1 \\ i\neq j}}^{n} d_i\alpha_i + \sum_{i=1}^{n}(d_jc_i)\alpha_i = \theta$

or, $\sum_{\substack{i=1 \\ i\neq j}}^{n} (d_i + d_jc_i)\alpha_i + (d_jc_j)\alpha_j = \theta$

As $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is linearly independent, so,

$$d_i + d_jc_i = 0, \text{ for } i = 1, 2, \ldots, j-1, j+1, \ldots, n \qquad — (1)$$

and $d_jc_j = 0$. As $c_j \neq 0$, $d_j = 0$

So, from (1), $d_i = 0$ for $i = 1, 2, \ldots, j-1, j+1, \ldots, n$

So, $d_i = 0$, for $i = 1, 2, \ldots, n$

Hence $\{\alpha_1, \alpha_2, \ldots \alpha_{j-1}, \beta, \alpha_{j+1}, \ldots, \alpha_n\}$ is linearly independent

We now prove that $L(T) = V$ where

$T = \{\alpha_1, \alpha_2, \ldots \alpha_{j-1}, \beta, \alpha_{j+1}, \ldots \alpha_n\}$ and let

$S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ Now $L(S) = V$

As every element of T is a linear combination of elements of S, so, $L(T) \subset L(S)$

Since α_j is a linear combination of elements of T (from (1)), so, each element of ~~T~~ S is a linear combination of elements of T. So, $L(S) \subset L(T)$

Hence $L(T) = L(S) = V$.

So, $\{\alpha_1, \alpha_2, \ldots, \alpha_{j-1}, \beta, \alpha_{j+1}, \ldots, \alpha_n\}$ is a basis of V.

Theorem 2.2.3. If $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of a finite dimensional vector space V over a field F, then ~~any~~ any linear independent set of vectors in V contains at most n vectors.

Proof: Let $\{\beta_1, \beta_2, \ldots, \beta_r\}$ be a linearly independent set of vectors in V. As $\{\beta_1, \ldots, \beta_r\}$ is linearly independent, $\beta_i \neq \theta$, for $i = 1, 2, \ldots, r$. Since $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis and $\beta_1 \neq \theta$ in V, $\beta_1 = \sum_{i=1}^{n} c_i \alpha_i$ where $c_i \in F$, $i = 1, 2, \ldots, n$, not all of which are zero. Let $c_i \neq 0$.

By Replacement Theorem, $\{\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \beta_1, \alpha_{i+1}, \ldots, \alpha_n\}$ is a basis for V. Since $\beta_2 \neq 0$ and $\{\alpha_1, \alpha_2, \ldots \alpha_{i-1}, \beta_1, \alpha_{i+1}, \ldots, \alpha_n\}$ is a basis of V, so $\beta_2 = \sum_{\substack{j=1 \\ j \neq i}}^{n} d_j \alpha_j + d_i \beta_1$ where $d_i \in F$, $i = 1, \ldots n$, not all of which are zero. We assert that at least one of $d_1, d_2, \ldots d_{i-1}, d_{i+1}, \ldots, d_n$ is non-zero. Because, if all of them be zero, then $\beta_2 = d_i \beta_1$. This implies $\{\beta_1, \beta_2\} \subset \{\beta_1, \ldots \beta_r\}$ is linearly dependent, a contradiction. So, let $d_j \neq 0$, $j \neq i$.

By Replacement Theorem $\{\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \beta_1, \alpha_{i+1}, \ldots, \alpha_{j-1}, \beta_2, \alpha_{j+1}, \ldots, \alpha_n\}$ is a basis of V. As $\beta_3 \neq 0$, $\beta_3 = \sum_{\substack{k \neq i \\ k \neq j}} t_k \alpha_k + t_i \beta_1 + t_j \beta_2$ where $t_k \in F$, $k = 1, \ldots, n$, not all zero.

we assert that at least one of $t_1, \ldots t_{i-1}, t_{i+1}, \ldots, t_{j-1}, t_{j+1}, \ldots t_n$ is non-zero. Because otherwise $\beta_3 = t_i \beta_1 + t_j \beta_2$ and this implies $\{\beta_1, \beta_2, \beta_3\} \subset \{\beta_1 \ldots \beta_r\}$, is linearly dependent, a contradiction.

Proceeding in this way, we observe that at each step one α is replaced by one β and the