$\beta = \{v_1, v_2, \ldots, v_n\}$ for V. We claim that

$S = \{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}$ is a basis for $R(T)$.

First we show that S generates $R(T)$, that is,

$$R(T) = span(S) (= L(S))$$

Using Theorem 3.1.3 and the fact that $T(v_i) = \theta'$ for $i = 1, 2, \ldots, k$, we have

$$R(T) = span(\{T(v_1), T(v_2), \ldots, T(v_n)\})$$
$$= span(\{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\})$$
$$= span(S)$$

Now we prove that S is linearly independent.

Suppose that $\sum_{i=k+1}^{n} b_i T(v_i) = \theta$ for $b_{k+1}, b_{k+2}, \ldots, b_n \in F$

Using the fact that T is linear, we have

$T\left(\sum_{i=k+1}^{n} b_i v_i\right) = \theta'$. So, $\sum_{i=k+1}^{n} b_i v_i \in N(T)$

Hence $\exists\ c_1, c_2, \ldots, c_k \in F$ such that

$$\sum_{i=k+1}^{n} b_i v_i = \sum_{i=1}^{k} c_i v_i \quad \text{or,} \quad \sum_{i=1}^{k} (-c_i) v_i + \sum_{i=k+1}^{n} b_i v_i = \theta$$

Since, β is a basis for V, we have

$b_i = 0$ for all $i = k+1, \ldots, n$. Hence S is linearly independent. So, $rank(T) = n - k$.

Hence $rank(T) + nullity(T) = dim(V)$.

If we apply the dimension Theorem to the linear transformation T in Example 9, we have $nullity(T) + 2 = 3$, so, $nullity(T) = 1$

Theorem 3.1.6 Let V and W be vector spaces, and let $T: V \to W$ be linear. Then T is one-to-one if and only if $N(T) = \{\theta\}$.

Proof: Suppose that T is one-to-one and $x \in N(T)$. Then $T(x) = \theta' = T(\theta)$. Since T is one-to-one, $x = \theta$. Hence $N(T) = \{\theta\}$. Now we assume that $N(T) = \{\theta\}$, and suppose that $T(x) = T(y)$. Then $\theta = T(x) - T(y) = T(x-y)$ by property 3 of Page-106. So, $x-y \in N(T) = \{\theta\}$. So, $x-y = \theta$, or $x = y$. This implies T is one-to-one.

Theorem 3.1.6 allows us to conclude that the transformation defined in Example 9 is not one-to-one.

Theorem 3.1.7 Let V and W be vector spaces of equal (finite) dimension, and let $T: V \to W$ be linear. Then the following are equivalent.

(a) T is one-to-one

(b) T is onto

(c) $\text{rank}(T) = \dim(V)$

Proof: From the dimension theorem, we have

$$\text{nullity}(T) + \text{rank}(T) = \dim(V).$$

Now, by the use of Theorem 3.1.6, we have that T is one-to-one if and only if $N(T) = \{\theta\}$, if and only if $\text{nullity}(T) = 0$, if and only if $\text{rank}(T) = \dim(V)$, if and only if $\text{rank}(T) = \dim(W)$, and if and only if

$\dim(R(T)) = \dim W$. As $R(T)$ is a subspace of W. So, $R(T) = W$, ~~Together~~ and this is the definition of T ~~to~~ being onto.

Example 11. Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be the linear transformation defined by $$T(f(x)) = 2f'(x) + \int_0^x 3f(t)\,dt$$

Now $R(T) = \text{span}(\{T(1), T(x), T(x^2)\}) = \text{span}(\{3x, 2+\frac{3}{2}x^2, 4x+x^3\})$.

Since $\{3x, 2+\frac{3}{2}x^2, 4x+x^3\}$ is linearly independent, $\text{rank}(T) = 3$. Since $\dim(P_3(\mathbb{R})) = 4$, T is not onto. From the dimension theorem, $\text{nullity}(T) + 3 = 3$. So, $\text{Nullity}(T) = 0$, and therefore, $N(T) = \{\theta\}$. We conclude from Theorem 3.1.6 that T is one-to-one.

Example 12 Let $T : F^2 \to F^2$ be the linear transformation defined by $T(a_1, a_2) = (a_1 + a_2, a_1)$

It is easy to see that $N(T) = \{\theta\}$ (check it.). So, T is one-to-one. Hence Theorem 3.1.7 tells us T must be onto.

Now we can prove this result: If T is linear and one-to-one then a subset S is linearly independent if and only if $T(S)$ is linearly independent. (Prove it.)

Example 13 illustrate the use of this result.

~~Exa~~ Example 13

Let $T : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformation defined by $T(a_0 + a_1 x + a_2 x^2) = (a_0, a_1, a_2)$.

Clearly, T is linear and one-to-one (Prove it).

Let $S = \{2-x+3x^2, x+x^2, 1-2x^2\}$. Then S is is linearly independent in $P_2(\mathbb{R})$ because $T(S) = \{(2,-1,3), (0,1,1), (1,0,-2)\}$ is linearly independent in $\mathbb{R}^3$.

One of the most ~~impo~~ important properties of a linear transformation is that it is completely determined by its action on a basis.

Theorem 3.1.8 Let V and W be vector spaces over F, and suppose that $\{v_1, v_2, \ldots, v_n\}$ is a basis for V. For $w_1, w_2, \ldots, w_n$ in W, there exists exactly one linear transformation $T: V \to W$ such that $T(v_i) = w_i$ for $i = 1, 2, \ldots, n$.

Proof: Let $x \in V$. Then

$x = \sum_{i=1}^{n} a_i v_i$, where $a_1, a_2, \ldots, a_n$ are unique scalars.

Define $T: V \to W$ by $T(x) = \sum_{i=1}^{n} a_i w_i$

(a) T is linear: Suppose that $u, v \in V$ and $d \in F$. Then we may write $u = \sum_{i=1}^{n} b_i v_i$ and $v = \sum_{i=1}^{n} c_i v_i$ for some scalars $b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n$. Thus

$du + v = \sum_{i=1}^{n} (db_i + c_i) v_i$. So,

$$T(du+v) = \sum_{i=1}^{n} (db_i + c_i) w_i = d\sum_{i=1}^{n} b_i w_i + \sum_{i=1}^{n} c_i w_i = dT(u) + T(v).$$