resulting set remains a basis of V. The following cases may arise:

(i) $\beta_1, \beta_2, \ldots \beta_r$ all come to the ~~basis~~ new basis containing some α's. In this case $r < n$

(ii) $\beta_1, \ldots, \beta_r$ exhaust all α's and form a new basis. In this case $r = n$

It can not happen that $r > n$. Because, then by replacement theorem, n vectors $\beta_1, \beta_2, \ldots \beta_n$ will come to the basis replacing all α's one after another and $\{\beta_1, \ldots, \beta_n\}$ becomes a new basis of V. So, the remaining vectors, $\beta_{n+1}, \beta_{n+2}, \ldots, \beta_r$ of V will each be a linear combination of $\beta_1, \ldots, \beta_n$ showing that the set $\{\beta_1, \beta_2, \ldots \beta_n, \beta_{n+1}, \ldots, \beta_r\}$ is linearly dependent, a contradiction.

So, $r \le n$. This complete the proof

~~Theorem~~ Corollary 2.2.4 Any two bases of a finite dimensional vector space V have the same number of vectors.

Proof: Let $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \beta_2, \ldots, \beta_m\}$ be two bases of ~~V~~ a finite dimensional vector space V. Since $\{\alpha_1, \ldots, \alpha_n\}$ is a basis of V and $\{\beta_1, \ldots, \beta_m\}$ is a linearly independent set of vectors in V, so, $m \le n$. Since $\{\beta_1, \ldots, \beta_m\}$ is a basis of V and $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ is linearly independent in V, so $n \le m$

Hence $n = m$.

Definition 2.2.5. The number of vectors in a basis of a vector space V is said to be the dimension of V and is denoted by $\dim V$. The null space $\{\theta\}$ is said to be of dimension zero.

Examples 1. $\dim \mathbb{R}^2 = 2$ as $E = \{(1,0),(0,1)\}$ is a basis of $\mathbb{R}^2$

The dimension of $\mathbb{R}^n$ is n as

$E = \{(1,0,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,0,\ldots,0,1)\}$ forms a basis of $\mathbb{R}^n$

2. The dimension of the Vector space $M_{m\times n}(\mathbb{R})$ is of all $m\times n$ real matrices is mn since the set

$\{E_{ij} : i=1,\ldots m, j=1,\ldots n\}$ forms a basis of $M_{m\times n}(\mathbb{R})$ where E_{ij} is an $m\times n$ matrix having 1 as the ijth element and all other elements are 0.

3. The dimension of $P_n(\mathbb{R})$, ~~of~~ the vector space of all real polynomials of degree at most n, is $n+1$ as $\{1, x, x^2, \ldots x^n\}$ forms a basis of $P_n(\mathbb{R})$

Theorem 2.2.6 Let V be a vector space of dimension n over a field F. Then ~~as~~ any linearly independent set of n vectors is a basis of V

Proof: Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a linearly independent set of vectors in V. Let β be an arbitrary vector in V and $\beta \neq \alpha_i$, $i=1,2,\ldots,n$. Since $\dim V = n$, any basis of V contains n vectors and the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta\}$ which contains $n+1$ vectors is linearly dependent. So, $\exists$ scalars $c_1, c_2, \ldots, c_n, c$ not all zero, in F such that

$$\sum_{i=1}^{n} c_i \alpha_i + c\beta = \theta \quad \ldots (i)$$

We assert that $c \neq 0$.

Because, $c=0$ implies $\sum_{i=1}^{n} c_i \alpha_i = \theta$ where $c_1, c_2, \ldots, c_n$ are not all zero, this implies the linear dependence of $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, a contradiction.

Since $c \neq 0$, $c^{-1} \in F$ and $c^{-1}c = 1$, 1 being the unity in F.

From (i), $\beta = \sum_{i=1}^{n} (-c^{-1}c_i)\alpha_i = \sum_{i=1}^{n} d_i \alpha_i$ where

$d_i = -c^{-1}c_i$, $i=1,\ldots,n$

This shows that β is a linear combination of $\alpha_1, \alpha_2, \ldots \alpha_n$. If however $\beta = \alpha_i$, for some i, then β is a linear combination of $\alpha_1, \alpha_2, \ldots \alpha_n$

Hence $L(\{\alpha_1, \alpha_2, \ldots, \alpha_n\}) = V$

So, $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a basis of V.

Theorem 2.2.7 Let V be a vector space of dimension n over a field F. Then any subset of n vectors of V that generates V, is a basis of V.

Proof: Since $\dim V = n$, any basis of V contains n vectors. Let $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a generating set of V, i.e., $L(S) = V$. We prove that S is linearly independent in V.

If S be linearly dependent, then by deletion theorem, $\exists$ a proper subset S_1 of S such that S_1 is linearly independent in V and $L(S_1) = V$.

So, S_1 is a basis of V containing less than n vectors of V and that contradicts the fact that $\dim V = n$.

So, S is linearly independent. So, S is a basis of V as $L(S) = V$.

<u>Theorem 2.2.8</u> Let S be a linearly independent subset of a vector space V, and let v be a vector in V that is not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in L(S)$

Proof: If $S \cup \{v\}$ is linearly dependent then there are vectors $u_1, u_2, \dots, u_n$ in $S \cup \{v\}$ such that

$\sum_{i=1}^{n} c_i u_i = \theta$ for non-zero scalars $c_1, c_2, \dots, c_n$ ~~not all zero~~

Because S is linearly independent, one of the u_i's, say u_1, equals v. So, $c_1 v + \sum_{i=2}^{n} c_i u_i = \theta$.

So, $v = \sum_{i=2}^{n} -(c_1^{-1} c_i) u_i$ as $c_1 \neq 0$ (In fact all of $c_1, c_2, \dots, c_n$ are non-zero.)