If $a=0, b\neq 0$, then the subspace is geometrically the y-axis and if $a\neq 0, b\neq 0$, then the subspace is a line passing through the origin, other than x and y axes.

Similarly, the subspaces of $\mathbb{R}^3$ are of dimension 0, 1, 2, and 3. origin is the subspace of dimension zero, the whole euclidean space of three dimension is the subspace $\mathbb{R}^3$ of dimension 3. All the lines passing through the origin are subspaces of dimension 1 and all the planes passing through origin are subspaces of dimension 2.

3.1 Linear Transformations, Null Spaces, Ranges

Definition 3.1.1 (Linear Transformation) Let V and W be vector spaces over a field F. We call a function $T: V \to W$ a linear transformation from V to W, if, for all $x, y \in V$ and $c \in F$, we have

(a) $T(x+y) = T(x) + T(y)$

(b) $T(cx) = cT(x)$

We often simply call T is linear.

The student should verify the following properties of a function $T: V \to W$:

1. If T is linear then $T(\theta) = \theta'$, the null element of W.

2. T is linear if and only if $T(cx+y) = cT(x)+T(y)$ for all $x, y \in V$ and $c \in F$.

3. If T is linear then $T(x-y) = T(x) - T(y)$ for all $x, y \in V$

4. T is linear if and only if, for $x_1, x_2, \ldots, x_n \in V$ and $c_1, c_2, \ldots, c_n \in F$,
$$T\left(\sum_{i=1}^{n} c_i x_i\right) = \sum_{i=1}^{n} c_i T(x_i)$$

We generally use the property 2 to prove that a given transformation is linear.

Example 1 Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a_1, a_2) = (2a_1+a_2, a_1)$

To show that T is linear, let $c \in \mathbb{R}$ and $x, y \in \mathbb{R}^2$, where $x = (b_1, b_2)$ and $y = (d_1, d_2)$

Since $cx+y = (cb_1+d_1, cb_2+d_2)$.

We have $T(cx+y) = \left(2(cb_1+d_1) + cb_2+d_2, cb_1+d_1\right)$
$$= \left(2cb_1 + 2d_1 + cb_2 + d_2, cb_1 + d_1\right)$$

Also $cT(x) + T(y) = c\left(2b_1+b_2, b_1\right) + \left(2d_1+d_2, d_1\right)$
$$= \left(c(2b_1+b_2), cb_1\right) + \left(2d_1+d_2, d_1\right)$$
$$= \left(2cb_1+cb_2, cb_1\right) + \left(2d_1+d_2, d_1\right)$$
$$= \left(2cb_1+cb_2+2d_1+d_2, cb_1+d_1\right)$$
$$= \left(2cb_1+2d_1+cb_2+d_2, cb_1+d_1\right)$$

Hence $T(cx+y) = cT(x)+T(y)$. So, T is linear.

Example 2 For any angle α, define $T_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ by the rule: $T_\alpha(a_1, a_2)$ is the vector obtained by rotating (a_1, a_2) counterclockwise by α if $(a_1, a_2) \neq (0, 0)$, and $T_\alpha(0,0) = (0,0)$. Then T_α is a linear transformation that is called rotation by α.

We determine an explicit formula for T_α.
Fix a non-zero vector $(a_1, a_2) \in \mathbb{R}^2$. Let β be the angle that (a_1, a_2) makes with the positive x-axis (see figure-1)

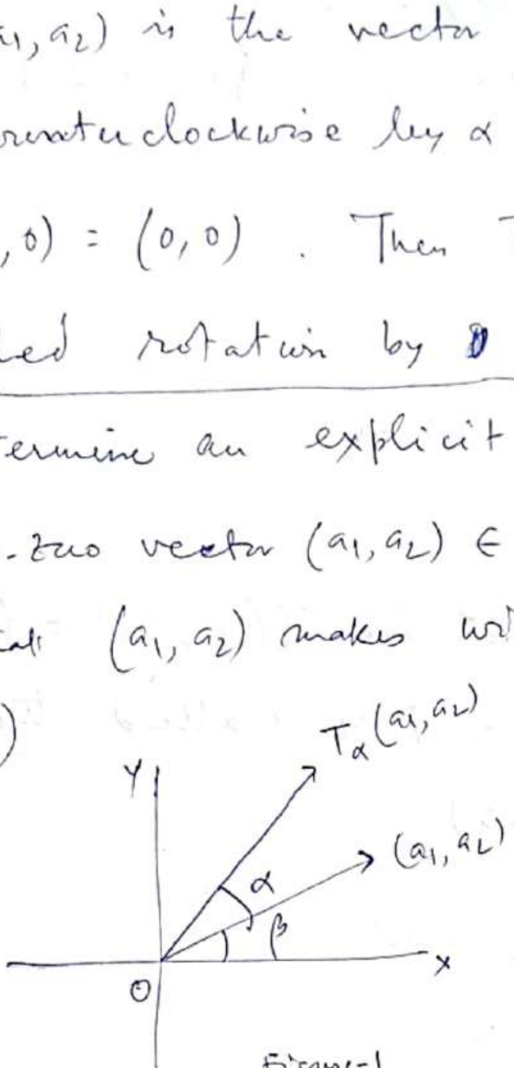

Figure-1

Let $r = \sqrt{a_1^2 + a_2^2}$. Then $a_1 = r\cos\beta$, $a_2 = r\sin\beta$

Also $T_\alpha(a_1, a_2)$ has length r and makes an angle $\beta+\alpha$ with the positive x-axis. It follows that

$$T_\alpha(a_1, a_2) = \left(r\cos(\beta+\alpha),\ r\sin(\beta+\alpha)\right)$$
$$= \left(r\cos\beta\cos\alpha - r\sin\beta\sin\alpha,\ r\sin\beta\cos\alpha + r\cos\beta\sin\alpha\right)$$
$$= \left(a_1\cos\alpha - a_2\sin\alpha,\ a_1\sin\alpha + a_2\cos\alpha\right)$$

Finally, observe that same formula is valid for $(a_1, a_2) = (0,0)$. Now we can easily check (check it) that T is linear.

Example 3 Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a_1, a_2) = (a_1, -a_2)$. Here T is linear (check it). T is called the reflection about the x-axis (see figure-2)

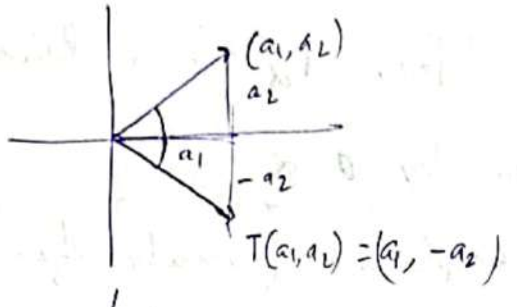

figure-2

Example 4 Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a_1, a_2) = (a_1, 0)$. T is linear (check it). T is called the projection on x-axis (see figure-3)

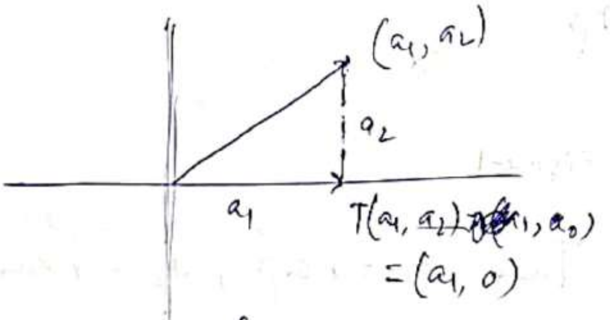

figure-3

Example 5 Define $T: M_{m\times n}(F) \to M_{n\times m}(F)$ by $T(A) = A^t$, where A^t is the transpose of A. Here T is linear as $T(cA+B) = (cA+B)^t = (cA)^t + B^t$

$= cA^t + B^t = cT(A) + T(B)$ for $c \in F$, $A, B \in M_{m\times n}(F)$

(as $(A+B)^t = A^t + B^t$ and $(cA)^t = cA^t$)

Example 6 Define $T: P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ by $T(f(x)) = f'(x)$ where $f'(x)$ denotes the derivative of $f(x)$.