To show that T is linear, let $g(x), h(x) \in P_n(\mathbb{R})$ and $a \in \mathbb{R}$.

Now $T(ag(x)+h(x)) = (ag(x)+h(x))' = ag'(x)+h'(x) = aT(g(x))+T(h(x))$

So, by property 2 above, T is linear.

Example 7 Let $V = C(\mathbb{R})$, the vector space of all continuous real valued function on $\mathbb{R}$. Let $a, b \in \mathbb{R}$, $a<b$

Define $T: V \to \mathbb{R}$ by $T(f) = \int_a^b f(t)\,dt$

for all $f \in V$. Then T is a linear transformation because the definite integral of a linear combination of functions is the same as the linear combination the definite integrals of functions.

Two very important examples of linear transformation that appear frequently in the remainder of the course, and therefore deserves their own notation, are identity and zero transformations.

For vector spaces V and W (over F), we define the identity transformation $I_V: V \to V$ by $I_V(x) = x$ for all $x \in V$ and the zero transformation $T_\theta: V \to W$ by $T_\theta(x) = \theta'$ for all $x \in V$, θ' is the null vector in W. It is clear that both of these transformations are linear (prove it).

We often write I instead of I_V.

We now turn our attention to two very important sets associated with linear transformations: the range and null space. The determination of these sets allows us to examine more ~~closely~~ closely the intrinsic properties of a linear transformation.

Definition: 3.1.1 Let V and W be vector space over a field F and let $T: V \to W$ be linear. We define the null space (or Kernel) $N(T)$ of T to be the set of all vectors x in V such that $T(x) = \theta'$; that is,

$$N(T) = \{x \in V : T(x) = \theta'\}.$$

We define the range (or image) $R(T)$ of T to be the subset of W consisting of all images (under T) of vectors in V; that is, $R(T) = \{T(x) : x \in V\}$.

Example 8 Let V and W be vector spaces over a field F, and let $I: V \to V$ and $T_\theta: V \to W$ be the identity and zero transformations, respectively. Then $N(I) = \{\theta\}$, $R(I) = V$, $N(I_\theta) = V$ and $R(I_\theta) = \{\theta'\}$

Example 9 Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

~~It~~ Verify that (Exercise) $N(T) = \{(a, a, 0) : a \in \mathbb{R}\}$ and $R(T) = \mathbb{R}^2$.

In examples 8 and 9, we see that the range and null space of each of the linear transformations is a subspace. The next result shows that this is true in general.

Theorem 3.1.2 Let V and W be vector spaces, and let $T: V \to W$ be linear. ~~If $\beta = \{v_1, v_2, \ldots, v_n\}$ is a~~

Lemma: Then $N(T)$ and $R(T)$ are subspaces of V and W respectively.

Proof: Since $T(\theta) = \theta'$, we have $\theta \in N(T)$. Let $x, y \in N(T)$ and $c \in F$. Then $T(x+y) = T(x) + T(y) = \theta' + \theta' = \theta'$ and $T(cx) = cT(x) = c\theta' = \theta'$. Hence $x+y \in N(T)$ and $cx \in N(T)$, so that $N(T)$ is a subspace of V.

Because, $T(\theta) = \theta'$, we have that $\theta' \in R(T)$. Now let $x, y \in R(T)$ and $c \in F$. Then $\exists$ u, v in V such that $T(u) = x$ and $T(v) = y$. So, $T(u+v) = T(u) + T(v) = x + y$. So, $x+y \in R(T)$ and $T(cu) = cT(u) = cx$. So, $cx \in R(T)$.

Hence $R(T)$ is a subspace of W

The next theorem provides a method for finding a spanning or generating set for the range of a linear transformation.

<u>Theorem 3.1.3</u>. Let V and W be vector spaces over a field F. If $\beta = \{v_1, v_2, \dots, v_n\}$ be a basis for V, then $R(T) = \text{span}\left(\{T(v_1), T(v_2), \dots, T(v_n)\}\right)$

Proof: Clearly $T(v_i) \in R(T)$ for each i. Because $R(T)$ is a subspace, $R(T)$ contains $\text{Span}\left(\{T(v_1), T(v_2), \dots, T(v_n)\}\right)$

Now suppose that $w \in R(T)$. Then $w = T(v)$ for some $v \in V$. Because β is a basis for V. So,

$$v = \sum_{i=1}^{n} a_i v_i \quad \text{for some } a_1, a_2, \dots, a_n \in F$$

Since T is linear, it follows that

$$w = T(v) = T\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1}^{n} a_i T(v_i)$$

So, $w \in span(\{T(v_1), T(v_2), \ldots, T(v_n)\})$

So, $R(T)$ is contained in $span(\{T(v_1), T(v_2), \ldots, T(v_n)\})$

So, $R(T) = span(\{T(v_1), T(v_2), \ldots T(v_n)\})$

Example 10 Define the linear transformation $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by $T(f(x)) = \begin{pmatrix} f(1)-f(2) & 0 \\ 0 & f(0) \end{pmatrix}$ (check for linearity)

Since $\beta = \{1, x, x^2\}$ is a basis for $P_2(\mathbb{R})$, we have

$$R(T) = span(\{T(1), T(x), T(x^2)\})$$
$$= span\left(\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}\right\}\right)$$
$$= span\left(\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\right\}\right)$$

So, we have found a basis for $R(T)$ and hence $\dim(R(T)) = 2$

Definition 3.1.4 Let V and W be vector spaces and $T: V \to W$ be linear. If $N(T)$ and $R(T)$ are finite dimensional, then we define the nullity of T, denoted nullity(T), and the rank of T, denoted rank(T), to be the dimension of $N(T)$ and $R(T)$, respectively.

Theorem 3.1.5 (Dimension Theorem) Let V and W be vector spaces, and let $T: V \to W$ be linear. If V is finite dimensional, then nullity(T) + rank(T) = $\dim(V)$.

Proof: Suppose that $\dim(V) = n$, $\dim(N(T)) = k$ and $\{v_1, v_2, \ldots, v_k\}$ is a basis for $N(T)$. So, by Extension Theorem, we may extend $\{v_1, v_2, \ldots, v_k\}$ to a basis