(b) Clearly $T(v_i) = w_i$ for $i = 1, 2, \ldots, n$.

(c) T is unique: Suppose that $U : V \to W$ is linear and $U(v_i) = w_i$ for $i = 1, 2, \ldots, n$. Then for $x \in V$ with

$$x = \sum_{i=1}^{n} a_i v_i, \text{ we have}$$

$$U(x) = \sum_{i=1}^{n} a_i U(v_i) = \sum_{i=1}^{n} a_i w_i = T(x).$$

Hence $U = T$.

Corollary 3.1.9 Let V and W be vector spaces, and suppose that V has a finite basis $\{v_1, v_2, \ldots, v_n\}$. If $U, T : V \to W$ are linear and $U(v_i) = T(v_i)$ for $i = 1, 2, \ldots, n$, then $U = T$.

Example 14 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(a_1, a_2) = (2a_2 - a_1, 3a_1)$, and suppose that $U : \mathbb{R}^2 \to \mathbb{R}^2$ is linear. If we know that $U(1,2) = (3,3)$ and $U(1,1) = (1,3)$, then $U = T$. This follows from corollary and from the fact that $\{(1,2), (1,1)\}$ is a basis for $\mathbb{R}^2$.

3.2 The matrix representation of a linear transformation

We first need the concept of an ordered basis for a vector space.

Definition 3.2.1 Let V be a finite dimensional vector space. An ordered basis for V is a basis for V endowed with a specific order; that is, an ordered basis for V is a finite sequence of linearly independent in V that generates V.

Example 1 ~~If~~ In F^3, let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$. Then $\beta = \{e_1, e_2, e_3\}$ can be considered an ordered basis of F^3. Also $\gamma = \{e_2, e_1, e_3\}$ is an ordered basis of F^3, but $\beta \neq \gamma$ as ordered bases.

For the vector space F^n, we call $\{e_1, e_2, \dots, e_n\}$ the standard ordered basis for F^n. Similarly, for the vector space $P_n(F)$, we call $\{1, x, x^2, \dots, x^n\}$ the standard ordered basis for $P_n(F)$.

Now that we have the concept of ordered basis, we can identify abstract vectors in an n dimensional vector space with ~~n-tuples~~ n-tuples. This identification is provided through the use of coordinate vectors, as introduced next.

Definition 3.2.2 Let $\beta = \{u_1, u_2, \dots, u_n\}$ be an ordered basis for a finite dimensional vector space V. For $x \in V$, let $a_1, a_2, \dots, a_n$ be the unique scalars such that

$$x = \sum_{i=1}^{n} a_i u_i$$

we define the coordinate vector of x relative to β, denoted by $[x]_\beta$, by $[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

notice that $[u_i]_\beta = e_i$ in the preceding definition. It is left as as an exercise ~~the~~ to show that the correspondence $x \to [x]_\beta$ provides us with a linear transformation from from V to F^n.

~~Exam~~ Example 2 Let $V = P_2(\mathbb{R})$, and let $\beta = \{1, x, x^2\}$ be the standard ordered basis for V. If $f(x) = 4+6x-7x^2$, then $[f]_\beta = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}$

Let us now proceed with the promised matrix representation of a linear transformation. Suppose that V and W are finite dimensional vector spaces with ordered bases $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_m\}$, respectively. Let $T: V \to W$ be linear. Then for each j, $j = 1, 2, \ldots, n$, there exist unique scalars $a_{ij} \in F$, $i = 1, 2, \ldots, m$, such that

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \quad \text{for } j = 1, 2, \ldots, n$$

Definition 3.2.3 Using the notation above, we call the $m \times n$ matrix A, defined by $A = [a_{ij}]_{m \times n}$, the matrix representation of T in the ordered bases β and γ and write $A = [T]_\beta^\gamma$. If $V = W$ and $\beta = \gamma$, then we write $A = [T]_\beta$.

Notice that the jth column of A is simply $[T(v_j)]_\gamma$.

Also observe that if $U: V \to W$ is a linear transformation such that $[U]_\beta^\gamma = [T]_\beta^\gamma$ then $U = T$ by corollary 3.1.9 of of Theorem 3.1.8.

We illustrate the computation of $[T]_\beta^\gamma$ in the next several examples.

Example 3 Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$.

Let β and γ be the standard ordered bases for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Now $T(1,0) = (1,0,2) = 1e_1 + 0e_2 + 2e_3$

and $T(0,1) = (3,0,-4) = 3e_1 + 0e_2 - 4e_3$

Hence $[T]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}$

If we let $\gamma' = \{e_3, e_2, e_1\}$ then

$$[T]_\beta^{\gamma'} = \begin{pmatrix} 2 & -4 \\ 0 & 0 \\ 1 & 3 \end{pmatrix}$$

Example 4 Let $T: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by $T(f(x)) = f'(x)$. Let β and γ be the standard ordered bases for $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$ respectively. Then

$$T(1) = 0.1 + 0.x + 0.x^2$$
$$T(x) = 1.1 + 0.x + 0.x^2$$
$$T(x^2) = 0.1 + 2.x + 0.x^2$$
$$T(x^3) = 0.1 + 0.x + 3.x^2$$

So, $[T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$. Note that when $T(x^j)$ is written as a linear combination of the vectors for γ, its coefficients