give the entries of the jth column $[T]_\beta^\gamma$.

Note: Let $T : V \to W$ be a linear transformation. Let $\beta = \{u_1, u_2, \ldots, u_n\}$ be an ordered basis for V and $\gamma = \{v_1, v_2, \ldots, v_m\}$ be an ordered basis for W and and $A = [a_{ij}]_{m\times n} = [T]_\beta^\gamma$. Then if $x \in V$ has coordinates $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ with respect to ordered basis β and $T(x)$ has the coordinates $\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$ with respect to the ordered basis γ. Then we have

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{or,} \quad [T(x)]_\gamma = A\,[x]_\beta$$

~~Definition 3.2.4 Let V and W be vector spaces over a field F, and let T, U : V → W~~

Definition 3.2.4 Let $T, U : V \to W$ be arbitrary functions, where V and W are vector spaces over the field F, and let $a \in F$. We define $T+U : V \to W$ by $(T+U)(x) = T(x) + U(x)$ for all $x \in V$ and $aT : V \to W$ by $(aT)(x) = aT(x)$ for all $x \in V$.

Theorem 3.2.5 Let V and W be vector spaces over a field F, and let $T, U : V \to W$ be linear. Then

(a) For all $a \in F$, $aT + U$ is linear

(b) Using the operations of addition and scalar multiplication in the preceeding definition, the collection of all linear transformations from V to W is a vector space over F.

Proof: (a) Let $x, y \in V$ and $c \in F$. Then

$$
\begin{aligned}
(aT+U)(cx+y) &= aT(cx+y) + U(cx+y) \\
&= a\left[T(cx+y)\right] + cU(x) + U(y) \\
&= a\left[cT(x) + T(y)\right] + cU(x) + U(y) \\
&= acT(x) + aT(y) + cU(x) + U(y) \\
&= c(aT+U)(x) + (aT+U)(y)
\end{aligned}
$$

So, $aT+U$ is linear.

(b) Noting that T_0, the zero transformation, plays the role of zero vector, it is easy to verify that the axioms of a vector space are satisfied, and hence that the collection of all linear transformations from V to W is a vector space over F.

Definition 3.2.6 Let V and W be vector spaces over a field F. We denote the vector space of all linear transformations from V to W by $L(V, W)$. In the case $V = W$, we write $L(V)$ instead of $L(V, W)$

Theorem 3.2.7 Let V and W be finite dimensional vector spaces with ordered bases β and γ, respectively, and let $T, U : V \to W$ be linear transformations, Then (a) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$ and

(b) $[aT]_\beta^\gamma = a[T]_\beta^\gamma$ for all scalars a.

Proof: Let $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_m\}$.

There exist unique scalars a_{ij} and b_{ij} $i=1,2,\ldots,m$, $j=1,2,\ldots,n$ such that

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \quad \text{and} \quad U(v_j) = \sum_{i=1}^{m} b_{ij} w_i \quad \text{for } j=1,2,\ldots,n$$

Hence $$(T+U)(v_j) = \sum_{j=1}^{m} (a_{ij} + b_{ij}) w_i \quad , j=1,2,\ldots,n$$

Then $[T+U]_\beta^\gamma = C = [c_{ij}]_{m\times n}$ and $[T]_\beta^\gamma = [a_{ij}]_{m\times n} = A$

and $[U]_\beta^\gamma = [b_{ij}]_{m\times n} = B$ then $c_{ij} = a_{ij} + b_{ij}$

So, $C = A+B$. So, $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

So (a) is proved, (b) is is proved similarly (Exercise).

Example 5 Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ and $U: \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformations respectively defined by

$T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$ and $U(a_1, a_2) = (a_1 - a_2, 2a_1, 3a_1 + 2a_2)$

Let β and γ be the standard ordered bases for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

Then $$[T]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}$$ and

$$[U]_\beta^\gamma = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{pmatrix}$$

If we compute $T+U$ using the preceding definition, we obtain $(T+U)(a_1, a_2) = (2a_1 + 2a_2, 2a_1, 5a_1 - 2a_2)$,

So, $$[T+U]_\beta^\gamma = \begin{pmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{pmatrix} = [T]_\beta^\gamma + [U]_\beta^\gamma,$$

illustrating Theorem 3.2.7.

3.3 Composition of two linear transformations.

We use more convenient notation UT rather than $U \circ T$ for the composite of linear transformations U and T.

Theorem 3.3.1 Let V, W, and Z be vector spaces over the same field F, and let $T: V \to W$ and $U: W \to Z$ be linear. Then $UT: V \to Z$ is linear.

Proof: Let $x, y \in V$ and $a \in F$. Then

$$UT(ax+y) = U(T(ax+y)) = U(aT(x) + T(y))$$
$$= aU(T(x)) + U(T(y)) = a(UT)(x) + UT(y)$$

So, UT is linear.

Theorem 3.3.2 Let V be a vector space over a field F. Let $T, U_1, U_2 \in L(V)$. Then

(a) $T(U_1 + U_2) = TU_1 + TU_2$

(b) $T(U_1 U_2) = (TU_1) U_2$

(c) $TI = IT = T$

(d) $a(U_1 U_2) = (aU_1) U_2 = U_1(aU_2)$ for all $a \in F$.

Proof: Exercise.