Let $T: V \to W$, $U: W \to Z$ be linear transformations, and let $A = [U]_{\beta}^{\gamma}$ and $B = [T]_{\alpha}^{\beta}$ where $\alpha = \{v_1, v_2, \ldots, v_n\}$, $\beta = \{w_1, w_2, \ldots, w_m\}$ and $\gamma = \{z_1, z_2, \ldots, z_p\}$ are ordered bases for V, W and Z, respectively.

We are going to find the matrix $[UT]_{\alpha}^{\gamma}$

For $j = 1, 2, \ldots, n$,

$$(UT)(v_j) = U(T(v_j)) = U\left(\sum_{k=1}^{m} b_{kj} w_k\right) = \sum_{k=1}^{m} b_{kj} U(w_k)$$

$$= \sum_{k=1}^{m} b_{kj} \left(\sum_{i=1}^{p} a_{ik} z_i\right) = \sum_{i=1}^{p} \left(\sum_{k=1}^{m} a_{ik} b_{kj}\right) z_i$$

$$= \sum_{i=1}^{p} c_{ij} z_i \quad \text{, where } A = [a_{ij}]_{p\times m} \quad B = [b_{ij}]_{m\times n}$$

and $\quad c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} \qquad i = 1, \ldots, p \quad j = 1, 2, \ldots, n$

So, $C = [c_{ij}]_{p\times n} = AB$

So, $[UT]_{\alpha}^{\gamma} = AB = [U]_{\beta}^{\gamma} [T]_{\alpha}^{\beta}$

This gives the Theorem 3.3.3 which is as follows:

Theorem 3.3.3. Let V, W and Z be finite dimensional vector spaces with ordered bases α, β and γ, respectively. Let $T: V \to W$ and $U: W \to Z$ be linear transformation,

Then $[UT]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma} [T]_{\alpha}^{\beta}$

Corollary 3.3.4 Let V be a finite dimensional vector space with an ordered basis β. Let $T, U \in L(V)$. Then $[UT]_{\beta} = [U]_{\beta} [T]_{\beta}$

Definition 3.3.5 Let A be an $m \times n$ matrix with entries from a field F. We denote L_A the mapping $L_A : F^n \to F^m$ defined by $L_A(x) = Ax$ (the matrix product of A and x) for each column vector $x \in F^n$. We called L_A a left-multiplication transformation.

Example 1 Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$. Then $A \in M_{2\times 3}(\mathbb{R})$ and

$L_A : \mathbb{R}^3 \to \mathbb{R}^2$. If $x = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

then $L_A(x) = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$

Theorem 3.3.6. Let A be an $m \times n$ matrix with entries from a field F. Then the left-multiplication transformation $L_A : F^n \to F^m$ is linear. Furthermore, if B is any other $m \times n$ matrix (with entries from F) and β and γ are the standard ordered bases of F^n and F^m, respectively, then we have the following properties:

(a) $[L_A]_\beta^\gamma = A$

(b) $L_A = L_B$ if and only if $A = B$

(c) $L_{A+B} = L_A + L_B$ and $L_{aA} = aL_A$ for all $a \in F$.

(d) If $T : F^n \to F^m$ is linear, then there exists a unique $m \times n$ matrix C such that $T = L_C$. In fact, $C = [T]_\beta^\gamma$

(e) If E is an $n \times p$ matrix, then $L_{AE} = L_A L_E$

(f) If $m = n$, then $L_{I_n} = I_{F^n}$

Proof: For $x, y \in F^n$, and $a \in F$, we have

$$L_A(ax+y) = A(ax+y) = a(Ax) + Ay = aL_A(x) + L_A y$$

So, L_A is linear.

(a) The jth column of $[L_A]_\beta^\gamma$ is equal to $L_A(e_j)$. However $L_A(e_j) = Ae_j$, which is also the jth column of A.

So, $[L_A]_\beta^\gamma = A$

(b) Here $L_A = L_B$. So, $A = [L_A]_\beta^\gamma = [L_B]_\beta^\gamma = B$ from (a).

and conversely if $A = B$ then $L_A(x) = Ax$ and $L_B(x) = Bx = Ax = L_A(x)$ as $A = B$. So, $L_A = L_B$.

(c) Let $x \in F^n$. Then $L_{A+B}(x) = (A+B)x = Ax + Bx$

$$= L_A(x) + L_B(x) = (L_A + L_B)(x)$$

So, $L_{A+B} = L_A + L_B$. Similarly, we can prove

$$L_{aA} = aL_A \text{ for all } a \in F.$$

(d) Let $C = [T]_\beta^\gamma$. Then we have, $[T(x)]_\gamma = [T]_\beta^\gamma [x]_\beta$

or, $T(x) = Cx = L_C(x)$ for all $x \in F^n$. Uniqueness of C follows from (b).

(e) and (f) are left as exercise.

3.4 Invertibility and Isomorphism.

Definition 3.4.1 Let V and W be vector spaces, and let $T: V \to W$ be linear. A function $U: V \to W$ is said to be an inverse of T if $TU = I_W$ and $UT = I_V$. If T has an inverse, then T is said to be invertible. We know that if T is invertible, then the inverse of T is unique and is denoted by T^{-1}

The following facts hold for invertible functions T and U;

1. $(TU)^{-1} = U^{-1}T^{-1}$
2. $(T^{-1})^{-1} = T$, in particular T^{-1} is invertible.

We often use the fact that a function is invertible if and only if it is both one-to-one and onto. We can therefore restate the Theorem 3.1.7 as follows:

3. Let $T: V \to W$ be a linear transformation, where V and W are finite dimensional vector spaces of equal dimension. Then T is invertible if and only if $\operatorname{rank}(T) = \dim(V)$

Example 1. Let $T: P_1(\mathbb{R}) \to \mathbb{R}^2$ be the linear transformation defined by $T(a+bx) = (a, a+b)$. The student can verify directly that $T^{-1}: \mathbb{R}^2 \to P_1(\mathbb{R})$ is defined by $T^{-1}(c,d) = c + (d-c)x$. Observe that T^{-1} is also linear. As Theorem 3-4.2 demonstrates, this is true in general.

Theorem 3.4.2 Let V and W be vector spaces, and let $T: V \to W$ be linear and invertible. Then $T^{-1}: W \to V$ is linear. Let $y_1, y_2 \in W$ and $c \in F$ (the scalar field). Since T is onto and one-to-one, there exists unique vectors x_1 and x_2 such that $T(x_1) = y_1$, $T(x_2) = y_2$