Then $x_1 = T^{-1}(y_1)$ and $x_2 = T^{-1}(y_2)$; So,

$$T^{-1}(cy_1+y_2) = T^{-1}(cT(x_1)+T(x_2)) = T^{-1}(T(cx_1+x_2)) = cx_1+x_2$$

$$= cT^{-1}(y_1) + T^{-1}(y_2)$$

now we develop some results that relate the inverses of the matrices to the inverses of linear transformations.

<u>Lemma 3.4.3</u> Let T be an invertible linear transformation from V to W. Then V is finite dimensional if and only if W is finite dimensional. In this case, $\dim(V) = \dim(W)$.

Proof: Suppose that V is finite dimensional. Let $\beta = \{x_1, x_2, \ldots, x_n\}$ be a basis for V. By Theorem 3.1.3 (Page-111) $T(\beta)$ spans $R(T) = W$; hence W is finite dimensional. Conversely, if W is finite dimensional, then so is V by similar argument using T^{-1}.

Now suppose that V and W are finite dimensional. Because T is ~~one~~ one-to-one and onto, we have

$$\text{nullity}(T) = 0 \quad \text{and} \quad \text{rank}(T) = \dim(R(T)) = \dim W$$

So, by dimension theorem (Page-112), it follows that $\dim(V) = \dim(W)$.

<u>Theorem 3.4.4</u> Let V and W be finite dimensional vector spaces with ordered bases β and γ, respectively. Let $T: V \to W$ be linear. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible. Furthermore, $[T^{-1}]_\beta^\gamma = ([T]_\beta^\gamma)^{-1}$

Proof: Suppose that T is invertible. By the Lemma 3.4.3, we have $\dim(V) = \dim(W)$. Let $n = \dim(V)$. So $[T]_\beta^\gamma$ is an $n \times n$ matrix. Now $T^{-1} : W \to V$ satisfies $TT^{-1} = I_W$ and $T^{-1}T = I_V$. Thus

$$I_n = [I_V]_\beta = [T^{-1}T]_\beta = [T^{-1}]_\gamma^\beta [T]_\beta^\gamma$$

Similarly, $[T]_\beta^\gamma [T^{-1}]_\gamma^\beta = I_n$. So, $[T]_\beta^\gamma$ is invertible and $\left([T]_\beta^\gamma\right)^{-1} = [T^{-1}]_\gamma^\beta$

Now suppose that $A = [T]_\beta^\gamma$ is invertible. Then there exists an $n \times n$ matrix B such that $AB = BA = I_n$

By Theorem 3.1.8, there exists $U \in L(W, V)$ such that

$$U(w_j) = \sum_{i=1}^{n} b_{ij} v_i \quad \text{for } j = 1, 2, \ldots, n$$

where $B = [b_{ij}]_{n \times n}$, $\beta = \{v_1, v_2, \ldots, v_n\}$, and $\gamma = \{w_1, w_2, \ldots, w_n\}$. It follows that $[U]_\gamma^\beta = B$

To show that $U = T^{-1}$, observe that

$$[UT]_\beta = [U]_\gamma^\beta [T]_\beta^\gamma = BA = I_n = [I_V]_\beta$$

By Theorem 3.3.3 (page - 125). So, $UT = I_V$ and similarly, $TU = I_W$

<u>Example 2</u> Let β and γ be the standard ordered bases for $P_1(\mathbb{R})$ and $\mathbb{R}^2$, respectively. For T as in example 1, we have $[T]_\beta^\gamma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $[T^{-1}]_\gamma^\beta = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

we can verify that $\left([T]_\beta^\gamma\right)^{-1} = [T^{-1}]_\gamma^\beta$.

Corollary 3.4.5 Let V be a finite dimensional vector space with an ordered basis β and let $T: V \to V$ be linear. Then T is invertible if and only if $[T]_\beta$ is invertible. Furthermore, $[T^{-1}]_\beta = ([T]_\beta)^{-1}$

Proof: Exercise

Corollary 3.4.6. Let A be an $n \times n$ matrix. Then A is invertible if and only if L_A is invertible. Furthermore, $(L_A)^{-1} = L_{A^{-1}}$.

Proof: Exercise.

Definition 3.4.7 Let V and W be vector spaces. We say that V is isomorphic to W if there exists a linear transformation $T: V \to W$ that is invertible. Such a linear transformation is called an isomorphism from V onto W.

Example 3 Define $T: F^2 \to P_1(F)$ by $T(a_1, a_2) = a_1 + a_2 x$. It can be easily checked that T is an isomorphism. So, F^2 isomorphic to $P_1(F)$

Theorem 3.4.8. Let V and W be finite dimensional vector spaces (over the same field F). Then V is isomorphic to W if and only if $\dim(V) = \dim(W)$.

Proof: Suppose that V is isomorphic to W and that $T: V \to W$ is an isomorphism from V to W. By the Lemma 3.4.3 of Theorem 3.4.4, we have that $\dim(V) = \dim(W)$.

Now suppose that $\dim(V) = \dim(W)$ and let $\beta = \{v_1, v_2, \dots, v_n\}$ and $\gamma = \{w_1, w_2, \dots, w_n\}$ be bases for V and W, respectively. By Theorem 3.1.8, there exists $T: V \to W$ such that T is linear and $T(v_i) = w_i$ for $i = 1, 2, \ldots, n$. Now, we have $R(T) = \text{span}(T(\beta)) = \text{span}(\gamma) = W$. So T is onto. From Theorem 3.1.7, we have that T is also one-to-one. Hence T is an isomorphism.

~~Cor.~~ Corollary 3.4.9 Let V be a vector space over the field F. Then V is isomorphic to F^n if and only if $\dim(V) = n$.

Theorem 3.4.9 Let V and W be finite dimensional vector space over the field F of dimensions n and m respectively, and let β and γ be ordered bases for V and W respectively. Then the function $\phi : L(V, W) \to M_{m\times n}(F)$, defined by $\phi(T) = [T]_\beta^\gamma$, is an isomorphism.

Proof: By Theorem 3.2.7, ϕ is linear. Hence we must show that ϕ is one-to-one and onto. This is acomplished if we show that for every $m\times n$ matrix A, there exists a unique linear transformation $T : V \to W$ such that $\phi(T) = A$. Let $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_m\}$ and let A be a given $m\times n$ matrix. By Theorem 3.1.8, there exists a unique linear transformation $T : V \to W$ such that

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \ , \ j = 1, 2, \ldots, n$$

But this means that $[T]_\beta^\gamma = A = [a_{ij}]_{m\times n}$ or, $\phi(T) = A$.

So, ϕ is an isomorphism.