Corollary 3.4.10 Let V and W are the finite dimensional vector spaces of dimension n and m respectively. Then $L(V, W)$ is finite dimensional of dimension mn.

Proof: The proof follows from Theorem 3.4.9 and Theorem 3.4.8 and the fact that $\dim(M_{m\times n}(F)) = mn$.

Definition 3.4.11 Let β be an ordered basis for an n dimensional vector space V over the field F. The standard representation of V with respect to β is the function $\phi_\beta : V \to F^n$ defined by $\phi_\beta(x) = [x]_\beta$ for each $x \in V$.

Example 4 Let $\beta = \{(1,0),(0,1)\}$ and $\gamma = \{(1,2),(3,4)\}$. It is easily observed that β and γ are ordered bases for $\mathbb{R}^2$. For $x = (1,-2)$, we have $\phi_\beta(x) = [x]_\beta = \begin{pmatrix}1\\-2\end{pmatrix}$ and $\phi_\gamma(x) = [x]_\gamma = \begin{pmatrix}-5\\2\end{pmatrix}$

We have observed earlier that ϕ_β is a linear transformation. The next theorem tells us much more.

Theorem 3.4.11 For any finite dimensional vector space V with ordered basis β, ϕ_β is an isomorphism.

Proof: Exercise

This theorem provides us with an alternative proof that an n dimensional vector space is isomorphic to F^n (see the corollary 3.4.9 of Theorem 3.4.8)

Let V and W be vector spaces of dimension n and m, respectively, and let $T: V \to W$ be a linear transformation. Define $A = [T]_\beta^\gamma$ where β and γ are ordered bases of V and W, respectively. We are now able to use ϕ_β and

and ϕ_γ to study the relationship between the linear transformations T and $L_A : F^n \to F^m$.

Let us first consider the figure 1

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \arrow[d, "\phi_\gamma"] \\ F^n \arrow[r, "L_A"'] & F^m \end{tikzcd}$$

Figure-1

Notice that there are two compositions of linear transformations that map V into F^m

1. Map V into F^n with ϕ_β and follow this transformation with L_A; this gives the composition $L_A \phi_\beta$

2. Map V into W with T and follow it by ϕ_γ to obtain the composition $\phi_\gamma T$

These two compositions depicted by Figure-1

Here we can show that $L_A \phi_\beta = \phi_\gamma T$, that is the diagram commutes.

Example 5 Let $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be linear transformation defined by $T(f(x)) = f'(x)$. Let β and γ be the standard ordered bases for $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$, respectively and let $\phi_\beta : P_3(\mathbb{R}) \to \mathbb{R}^4$ and $\phi_\gamma : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the corresponding standard representations of $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$. If

$A = [T]_\beta^\gamma$ then we have

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$. Consider the polynomial $p(x) = 2 + x - 3x^2 + 5x^3$.

We show that $L_A \phi_\beta(p(x)) = \phi_\gamma T(p(x))$.

Now, $L_A \phi_\beta(p(x)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ 15 \end{pmatrix}$

But since $T(p(x)) = p'(x) = 1 - 6x + 15x^2$, we have

$$\phi_\gamma T(p(x)) = \phi_\gamma(T(p(x))) = \phi_\gamma(1 - 6x + 15x^2) = \begin{pmatrix} 1 \\ -6 \\ 15 \end{pmatrix}$$

So, $L_A \phi_\beta(p(x)) = \phi_\gamma T(p(x))$

3.5 Eigen values, eigen vectors and characteristic equation of a matrix. Cayley Hamilton theorem and its use in finding the inverse of a matrix.

Definition 3.5.1:

(<u>Characteristic equation</u>) Let A be an $n \times n$ matrix over a field F. Then $\det(A - xI_n)$ is said to be the characteristic polynomial of A and is denoted by $\psi_A(x)$. The equation $\psi_A(x) = 0$ is said to be the characteristic equation of A.

Let $A = (a_{ij})_{n\times n}$. Then

$$\psi_A(x) = \begin{vmatrix} a_{11}-x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-x & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-x \end{vmatrix}$$

$= c_0 x^n + c_1 x^{n-1} + \cdots + c_n$ where $c_0 = (-1)^n$ and

$c_r = (-1)^{n-r}$ [sum of the principal minors of A of order r]

In particular, $c_1 = (-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn})$

$= (-1)^{n-1}$ trace A

and $c_n = \det A$.

The degree of the characteristic equation is same as the order of the matrix A and the coefficients are scalars belonging to F.

Theorem 3.5.2 (Cayley-Hamilton Theorem) Every square matrix satisfies its own characteristic equation.

The theorem states that if A be an $n\times n$ matrix and the characteristic polynomial of A be $c_0x^n+c_1x^{n-1}+\cdots+c_n$, then

$$c_0A^n+c_1A^{n-1}+\cdots+c_nI_n = O$$

Proof: Let A be a $n\times n$ matrix. Then $\det(A-xI) = c_0x^n+c_1x^{n-1}+\cdots+c_n$. $A-I_nx$ is a matric polynomial in x of degree 1 and $\operatorname{adj}(A-I_nx)$ is a matric polynomial in x of degree $n-1$, since each element of $\operatorname{adj}(A-xI_n)$ (i.e., a cofact of an element of the matrix $A-I_nx$) is a polynomial in x of degree $n-1$ at most.

Let $\operatorname{adj}(A-I_nx) = B_0x^{n-1}+B_1x^{n-2}+\cdots+B_{n-1}$, where each B_i is an $n\times n$ matrix.

$$(A-xI_n)\operatorname{adj}(A-xI_n) = \left(\det(A-xI_n)\right)I_n \quad \text{gives}$$

$$(A-xI_n)\left(B_0x^{n-1}+B_1x^{n-2}+\cdots+B_{n-1}\right) = \left(c_0x^n+c_1x^{n-1}+\cdots+c_n\right)I_n$$

or, $$A\left(B_0x^{n-1}+B_1x^{n-2}+\cdots+B_{n-1}\right)-\left(B_0x^n+B_1x^{n-1}+\cdots+B_{n-1}x\right)$$
$$= (c_0I_n)x^n+(c_1I_n)x^{n-1}+\cdots+c_nI_n$$

Equating the coefficients of like powers of x, we have

$$\left.\begin{aligned}
-B_0 &= c_0I_n\\
AB_0-B_1 &= c_1I_n\\
AB_1-B_2 &= c_2I_n\\
&\cdots\\
AB_{n-2}-B_{n-1} &= c_{n-1}I_n\\
AB_{n-1} &= c_nI
\end{aligned}\right\} \cdots (\mathrm{I})$$

premultiplying the relations (I) by $A^n, A^{n-1}, \ldots, A, I_n$ respectively and adding, we have

$$c_0A^n+c_1A^{n-1}+\cdots+c_{n-1}A+c_nI_n = O$$

This completes the proof.