Cayley-Hamilton Theorem gives a computational procedure for obtaining A^{-1} when A is a non-singular matrix.

Let the characteristic equation be $c_0x^n + c_1x^{n-1} + \cdots + c_n = 0$. By Cayley-Hamilton Theorem, $c_0A^n + c_1A^{n-1} + \cdots + c_{n-1}A + c_nI_n = O$

Since $c_n = \det A \neq 0$, c_n^{-1} exists in F. Multiplying by $-c_n^{-1}$, we have $-c_n^{-1}(c_0A^n + c_1A^{n-1} + \cdots + c_{n-1}A) - I_n = O$

or, $-c_n^{-1}(c_0A^{n} + c_1A^{n-1} + \cdots + c_{n-1}I_n)A = I_n$

From the definition and uniqueness of inverse, it follows that $A^{-1} = -c_n^{-1}(c_0A^n + c_1A^{n-1} + \cdots + c_{n-1}I_n)$

Thus A^{-1} is expressed as a polynomial in A with coefficients from the field F.

<u>Worked Examples</u>

1. Use Cayley-Hamilton Theorem to find A^{-1}, where $A = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$

Solution: The characteristic equation of A is

$$\begin{vmatrix} 2-x & 1 \\ 3 & 5-x \end{vmatrix} = 0 \quad \text{or, } x^2 - 7x + 7 = 0$$

Hence $\det A = 7 \neq 0$, A^{-1} exist.

By Cayley Hamilton theorem,

$$A^2 - 7A + 7I_2 = O$$

$$\text{or, } -\frac{1}{7}A(A - 7I_2) = I_2$$

This gives $A^{-1} = -\frac{1}{7}(A - 7I_2) = \frac{1}{7}\begin{pmatrix} 5 & -1 \\ -3 & 2 \end{pmatrix}$

2. Use Cayley Hamilton theorem to find A^{50}, where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Solution: The characteristic equation of A is

$x^2-2x+1=0$. So, by Cayley-Hamilton theorem, we have, $A^2-2A+I_2=O$ or, $A^2-A=A-I_2$. So, $A^3-A^2=A^2-A=A-I_2$, $\ldots$, $A^{50}-A^{49}=A-I_2$. Adding, we have

$$A^{50}-A=49A-49I_2 \quad \text{or, } A^{50}=50A-49I_2=\begin{pmatrix}1 & 50\\ 0 & 1\end{pmatrix}$$

Definition 3.5.3: (Eigen value of a matrix) A root of the characteristic equation of a square matrix A is said to be an eigenvalue of A.. Although the coefficients of $\psi_A(x)$ are elements of F, the eigenvalues of A may not be all elements of F. But they all belong to a suitable algebraic extension of the field F.

For example, if the ground field F of A be $\mathbb{R}$ then $\psi_A(x)$ is a real polynomial but the roots of $\psi_A(x)=0$ may not be all real. They are all elements of the field $\mathbb{C}$ of all complex numbers which is an algebraic extension of $\mathbb{R}$.

A root of $\psi_A(x)=0$ of multiplicity r is said to be an r-fold eigen value of A.

Theorem 3.5.4 The product of the eigen values of a square matrix A is det A.

Proof: Let A be an $n\times n$ matrix and let the characteristic equation of A be $c_0x^n+c_1x^{n-1}+\cdots+c_n=0$

Then $c_0=(-1)^n$, $c_n=\det A$

The product of the roots of the equation is $(-1)^n\frac{c_n}{c_0}=c_n=\det A$.

Theorem 3.5.5 If A be a singular matrix, then 0 is an eigenvalue of A

Proof: Since A is singular, $\det A=0$.

Let $\varphi_A(x) = c_0 x^n + c_1 x^{n-1} + \cdots + c_n$. Then $c_n = \det A = 0$
Consequently, 0 is a root of the characteristic equation of A and therefore 0 is an eigen value of A.

Theorem 3.5.6 The eigenvalues of a diagonal matrix are its diagonal elements

Proof: Let $A = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}$

Then $\det(A - xI_n) = (d_1 - x)(d_2 - x)\cdots(d_n - x)$
So, the roots of $\det(A - xI_n) = 0$ are $d_1, d_2, \ldots, d_n$
So, the eigenvalues of A are $d_1, d_2, \ldots, d_n$

Theorem 3.5.7 If λ be an r-fold eigenvalue of a square matrix A, 0 is an r-fold eigenvalue of the matrix $A - \lambda I_n$

Proof: Let $\varphi_A(x) = \det(A - xI_n) = (x - \lambda)^r \phi(x)$ where $\phi(x)$ is a polynomial of degree $n-r$ and $\phi(\lambda) \neq 0$
The characteristic polynomial of $A - \lambda I_n$ is

$$\begin{aligned}\det(A - \lambda I_n - xI_n) &= \det(A - (\lambda + x)I_n) \\ &= (\lambda + x - \lambda)^r \phi(\lambda + x) \\ &= x^r \mu(x), \text{ where } \mu(0) = \phi(\lambda) \neq 0\end{aligned}$$

This proves that 0 is an r-fold eigenvalue of $A - \lambda I_n$

Theorem 3.5.8 If λ be an eigen value of a non-singular matrix A, then λ^{-1} is an eigenvalue of A^{-1}

Proof: Let the order of A be n. Since A is non-singular, A^{-1} exists. Also $\lambda \neq 0$. So, λ^{-1} exists.
Since λ is an eigenvalue of A is $\det(A - \lambda I_n) = 0$

$$\det(A^{-1} - \lambda^{-1} I_n) = (\det A)^{-1} \det(A A^{-1} - \lambda^{-1} A)$$
$$= (\det A)^{-1} (\lambda^{-1})^n \det(\lambda I_n - A)$$
$$= (\det A)^{-1} (\lambda^{-1})^n (-1)^n \det(A - \lambda I_n)$$
$$= 0 \quad \text{since } \det(A - \lambda I_n) = 0$$

So, λ^{-1} is an eigenvalue of A^{-1}.

Theorem 3.5.9 If A and P be both $n \times n$ matrices and P be non-singular, then A and $P^{-1}AP$ have the same eigen value

Proof: $\det(P^{-1}AP - x I_n) = \det(P^{-1}AP - P^{-1}(x I_n) P)$
$$= \det(P^{-1}(A - x I_n) P) = \det P^{-1} \det(A - x I_n) \det P$$
$$= \det(A - x I_n) \det(P^{-1} P)$$
$$= \det(A - x I_n) \det(I_n)$$
$$= \det(A - x I_n)$$

So the characteristic polynomial of A and $P^{-1}AP$ are same. So, they have same eigenvalues.

Example Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{pmatrix}$

The characteristic equation of A is $\begin{vmatrix} 1-x & -1 & 0 \\ 1 & 2-x & -1 \\ 3 & 2 & -2-x \end{vmatrix} = 0$

or, $(1-x)(x^2-2) + (1-x) = 0$

or, $(1-x)(x^2-1) = 0$

So, the ~~eqn~~ eigen values of A are $1, 1, -1$

Definition 3.5.10 (Eigenvectors of a matrix):
Let A be $n \times n$ matrix over a field F. A non-null vector X belonging to $V_n(F)$ is said to be an eigen vector