A if there exists a scalar $\lambda \in F$ such that $AX = \lambda X$ holds. ($V_n(F)$ is a vector space of dimension over the field F)

Let X be an eigenvector of the matrix A. Then for some suitable scalar λ, $AX = \lambda X$ holds. That is, $(A - \lambda I_n)X = O$. This is a homogeneous system of n equations in n unknowns. Since there exists a non-null solution of the system, $\det(A - \lambda I_n) = 0$. This implies that λ is an eigenvalue of A. Thus for an eigenvector, if it exists, there corresponds an eigenvalue of A.

Theorem 3.5.11 Let A be an $n \times n$ matrix over a field F. To an eigenvector of A there corresponds a unique eigen value of A.

Proof: Let there be two distinct eigenvalues λ_1, λ_2 of A corresponding to an eigen vector X. Then $AX = \lambda_1 X$ and $AX = \lambda_2 X$. So, $(\lambda_1 - \lambda_2)X = O$. But this is a contradiction, since X is a non-null vector and $\lambda_1 \neq \lambda_2$. Hence the theorem.

Theorem 3.5.12 Let A be an $n \times n$ matrix over a field F and λ be an eigenvalue belonging to F. To each such eigenvalue of A there corresponds at least one eigen vector.

Proof: Since λ is an eigenvalue, $\det(A - \lambda I) = 0$. So, the homogeneous system of equations
$$(A - \lambda I_n)X = O$$
has a non-null solution,

say $X = X_1$ where $X_1 \in V_n(F)$. Then $(A - \lambda I_n) X_1 = 0$
or, $AX_1 = \lambda X_1$. This shows that X_1 is an eigen vector of A corresponding to λ. This proves the theorem.

Note: In fact, there are many eigen vectors of A corresponding to an eigen value $\lambda \in F$, because $\det(A - \lambda I_n) = 0$ implies that there are infinite number of non-null solutions of the homogeneous system $(A - \lambda I_n) X = 0$ and each such non-null solution gives an eigen vector of A corresponding to λ.

Example Let $A = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}$. The eigen values of A are $-1, 7$. Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be an eigen vector corresponding to -1. Then $AX = -X$ and this gives
$$2x_1 + 3x_2 = 0$$
$$4x_1 + 6x_2 = 0$$

The equivalent system is $x_1 + \frac{3}{2} x_2 = 0$

The solution of the system is $k(-\frac{3}{2}, 1)$ where $k \in \mathbb{R}$.

The eigen vectors are $k\begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$ or equivalently, $c\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, where c is a non-zero real number.

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be an eigen vector corresponding to 7. Then $AX = 7X$ and this gives
$$-6x_1 + 3x_2 = 0$$
$$4x_1 - 2x_2 = 0$$

The system is equivalent to $x_1 - \frac{1}{2} x_2 = 0$

The solution of the system is $c(1, 2)$, where $c \in \mathbb{R}$

Therefore the eigen vector are $c\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, where c is a non-zero real number.

Theorem 3.5.13 If X_i be the eigen vector corresponding to eigen value λ_i, $i=1,2,\ldots,k$ (where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all distinct) of the square matrix A over a field F, then ~~set~~ $\{X_1, X_2, \ldots, X_k\}$ is linearly independent.

Proof: $AX_i = \lambda_i X_i$, $i=1,2,\ldots,k$

Let us consider the relation $\sum_{i=1}^{k} c_i X_i = \theta$ —— (1)

So, $\sum_{i=1}^{k} c_i(AX_i) = \theta$

or, $$\sum_{i=1}^{k} c_i \lambda_i X_i = \theta \quad \text{——} \ (2)$$

Similarly $$\sum_{i=1}^{k} c_i A^2 X_i = \theta$$

or, $$\sum_{i=1}^{k} c_i \lambda_i^2 X_i = \theta \quad \text{- - - -} \ (3)$$

Similarly, $$\sum_{i=1}^{k} c_i \lambda_i^3 X_i = \theta \quad \text{- - -} \ (4)$$

- - - - -

$$\sum_{i=1}^{k} c_i \lambda_i^{k-1} X_i = \theta \quad \text{- - - -} \ (k)$$

So, we can write from equations (1) to (k),

$$(c_1X_1, c_2X_2, \ldots, c_kX_k) \begin{pmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \lambda_k & \cdots & \lambda_k^{k-1} \end{pmatrix} = (0,0,\ldots,0) \quad \text{—} \ (A)$$

Let $P = \begin{pmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{k-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{k-1} \\ \vdots & & & \\ 1 & \lambda_k & \cdots & \lambda_k^{k-1} \end{pmatrix}$ now $\det P = \prod_{i<j} (\lambda_i - \lambda_j) \neq 0$

as $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all distinct. So, P^{-1} exists

Premultiplying both sides of (A) by P^{-1}, we have

$$(c_1X_1, c_2X_2, \ldots, c_kX_k) = (0, 0, \ldots, 0)$$

So, $c_i = 0$, $i = 1, 2, \ldots, k$ since $X_1, X_2, \ldots, X_k$ are all non-zero.
This proves that $\{X_1, X_2, \ldots, X_k\}$ ~~are~~ is linearly dependent.

Theorem 3.5.14 The eigen vectors of an $n \times n$ matrix A over a field F corresponding to an eigen value $\lambda \in F$, together with the null vector, form a vector space, a subspace of $V_n(F)$.

Proof: To an eigen value $\lambda \in F$, there corresponds an eigen vector of A. Let S be the set of all eigen vectors of A corresponding to λ. Then S is the set of all non-null solutions of the homogeneous system of equations $(A - \lambda I_n)X = 0$. The null vector $\theta \in V_n(F)$ is also a solution of the system.

As the solutions of a homogeneous system with an $n \times n$ matrix over F as the coefficient matrix form a subspace of $V_n(F)$, it follows that $S \cup \{\theta\}$ is a subspace of $V_n(F)$.

In other words, the eigen vectors corresponding to λ, together with the null vector, form a non-null ~~sp~~ subspace of $V_n(F)$. This completes the proof.

Definition 3.5.15 The subspace $E_\lambda = \{x \in V_n(F) : Ax = \lambda x\}$ in Theorem 3.5.14 is called the eigenspace ~~corr~~ corresponding to an eigen value λ.

Theorem 3.5.16 If λ be an r-fold eigen value of an $n \times n$ matrix A, then rank of $(A - \lambda I_n) \geq n - r$

Proof: Since λ is an r-fold eigen value of A,