0 is an r-fold eigen value of $A - \lambda I_n$ by theorem 3.5.7

Let $\det(A - \lambda I_n - x I_n) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$

Then $a_0 = (-1)^n$, $a_k = (-1)^{n-k}$ [Sum of all principal minors of order k of $A - \lambda I_n$], for $k = 1, 2, \ldots, n$

Since 0 is an r-fold eigen value of $A - \lambda I_n$,

$a_n = a_{n-1} = \cdots = a_{n-r+1} = 0$, $a_{n-r} \neq 0$. So the sum of all principal minors of order $n-r$ of $A - \lambda I_n$ is not zero and hence there exists at least one non-zero minor of order $n-r$ of the matrix $A - \lambda I_n$

It follows that rank of $(A - \lambda I_n) \geq n-r$

Corollary 3.5.17 If λ be a simple eigen value (i.e., of multiplicity 1) of A, then rank of $(A - \lambda I_n) = n-1$

Because, by the theorem 3.5.16, rank of $(A - \lambda I_n) \geq n-1$ and since λ is an eigen value, $\det(A - \lambda I_n) = 0$ and so rank of $(A - \lambda I_n) \leq n-1$.

Theorem 3.5.18 If λ be an r-fold eigen value of A, the rank of the eigenspace corresponding to λ does not exceed r.

Proof: The ~~class~~ eigenspace is the subspace of solutions of the homogeneous system $(A - \lambda I_n) X = 0$

So, the rank of eigenspace + rank of $(A - \lambda I_n) = n$

Since rank of $(A - \lambda I_n) \geq n-r$, so,

the rank of eigenspace $\leq r$.

Corollary 3.5.19 The rank of the eigenspace corresponding to a simple eigenvalue λ is 1, since, in this case, rank of $(A - \lambda I_n) = n-1$

Definition 3.5.20 For an r-fold eigen value λ, r is called the algebraic multiplicity of λ and the rank of the eigenspace corresponding to λ is called the geometric multiplicity of λ. Since the eigenspace is always a non-null subspace, it follows that for an eigen value λ,

$$1 \le \text{geometric multiplicity} \le \text{algebraic multiplicity}.$$

An eigen value λ is said to be regular if the geometric multiplicity of λ is equal to its algebraic multiplicity.

Example Let $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

The characteristic equation of A is $x^2(1-x) = 0$ and therefore the eigen values of A are $0, 0, 1$. 0 is an eigen value of algebraic multiplicity 2, and 1 is a simple eigen value of A (i.e., of algebraic multiplicity 1).

The eigen vectors corresponding to the eigen value 0 are the non-null solutions of the system

$$x + y + z = 0$$
$$z = 0$$

The eigen vectors are $c\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, where c is a non-zero real number. The rank of the eigen space corresponding to eigen value 0 is 1. So, the geometric multiplicity of 0 is 1. So, in this case, the geometric multiplicity is less ~~than~~ than algebraic multiplicity.

The eigen vectors corresponding to the eigen value 1 are the non-null solutions of the system $y+z=0$
$x+2y+z=0$

The solution of the system is $c\begin{pmatrix}1\\-1\\1\end{pmatrix}$, where $c \in \mathbb{R}$.

So, the rank of the eigen space corresponding to eigen value 1 is 1. So, the geometric multiplicity of the eigen value is 1, Here geometric multiplicity = algebraic multiplicity and 1 is a regular eigen value.

Theorem 3.5.21 The eigen values of a real symmetric matrix are all real.

Proof: Let A be an $n \times n$ real symmetric matrix. The characteristic equation of A is an equation with real coefficients. So, the eigen values of A are complex numbers, some or all of which may be purely real.

Let λ be an eigen value of A. Then $\det(A-\lambda I_n) = 0$. So, $\exists$ non-null solutions of the homogeneous system $(A-\lambda I_n)X = 0$. Let X_1 be one such solution. Then $(A-\lambda I_n)X_1 = 0$. That is $AX_1 = \lambda X_1$.

[Note that this X_1 is not an eigen vector of A unless λ is purely real.]

Taking transpose of the conjugate, we have

$$\left(\overline{AX_1}\right)^t = \left(\overline{\lambda X_1}\right)^t$$

or, $\left(\overline{X_1}\right)^t\left(\overline{A}\right)^t = \overline{\lambda}\left(\overline{X_1}\right)^t$, since λ is a scalar

or, $\left(\overline{X_1}\right)^t A = \overline{\lambda}\left(\overline{X_1}\right)^t$, since $\overline{A}^t = A^t = A$

Multiplying by X_1 from right, we have

$$(\bar{X_1})^t A X_1 = \bar{\lambda}(\bar{X_1})^t X_1$$

or, $(\bar{X_1})^t \lambda X_1 = \bar{\lambda}(\bar{X_1})^t X_1$

or, $\lambda(\bar{X_1})^t X_1 = \bar{\lambda}(\bar{X_1})^t X_1$

or, $(\lambda - \bar{\lambda})(\bar{X_1})^t X_1 = 0$

But $(\bar{X_1})^t X_1 \neq 0$ as X_1 is non-null

It follows that $\lambda = \bar{\lambda}$ and so, λ is real.

Theorem 3.5.22 The eigen values of a real skew-symmetric matrix are purely imaginary or zero.

Proof: Let A be an $n \times n$ real skew symmetric matrix. Following the same argument as in the previous theorem, we have

$$(\lambda + \bar{\lambda})(\bar{X_1})^t X_1 = 0, \text{ since } \bar{A}^t = A^t = -A$$

Since X_1 is non-null, $\lambda + \bar{\lambda} = 0$. That is, $\lambda = -\bar{\lambda}$

Therefore λ is purely imaginary or zero and the theorem is proved.

Note 1, The eigen values of a Hermitian matrix are all real

Let λ be an eigen value of a Hermitian matrix A and X be an eigen vector corresponding to λ. Then $AX = \lambda X$. Using $\bar{A}^t = A$ and proceeding in a similar manner as in the Theorem 3.5.21, the assertion can be established

Note 2 The eigen values of a skew-Hermitian matrix are all purely imaginary or zero.